Linearity / Time Invariance

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Linearity

- A system is *linear* if and only if:

$$f(Ax) = Af(x)$$

- It is time time invariant if

$$y(t) = f(x(t)) \text{ and}$$

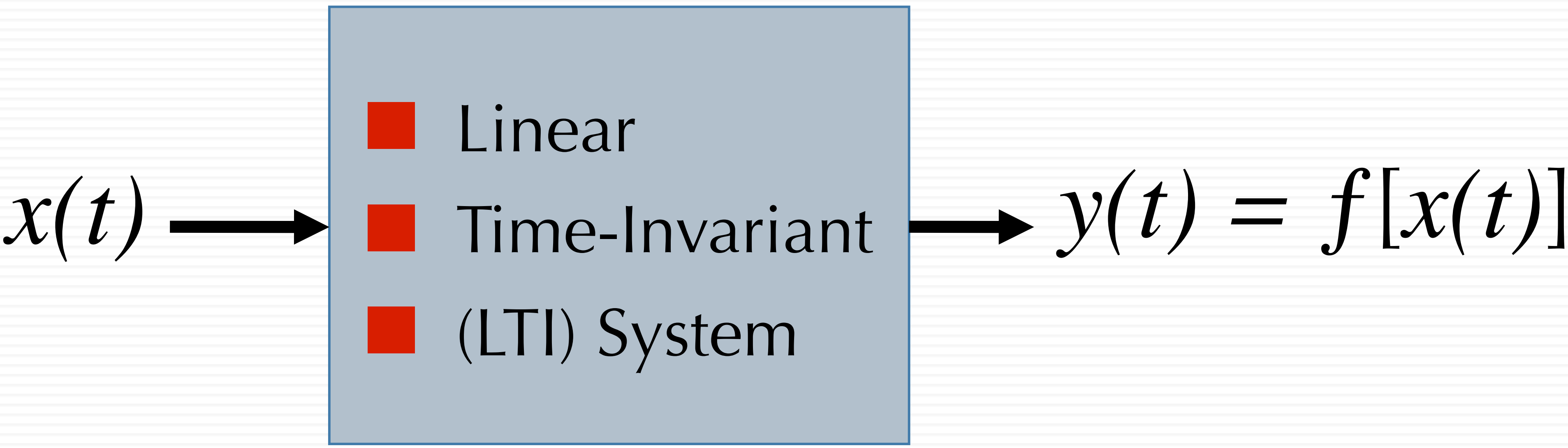
$$y(t+\delta) = f(x(t+\delta))$$

for all δ.

Linear Systems Approach

In an LTI system, given two inputs A & B:

$$f(A + B) = f(A) + f(B)$$



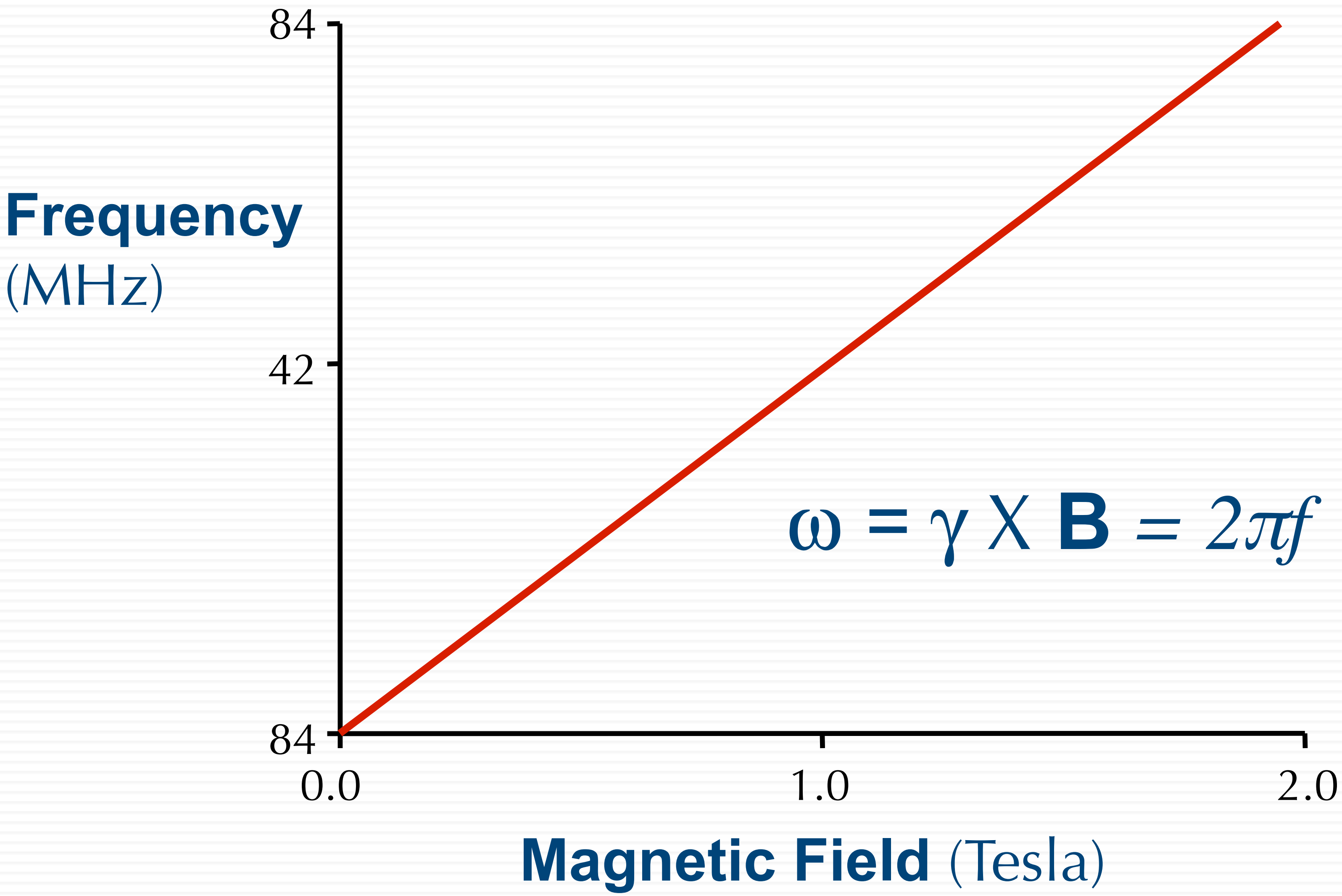
*Everything is Linear to First Order *Almost

"The faster you drive, the quicker you will get there."

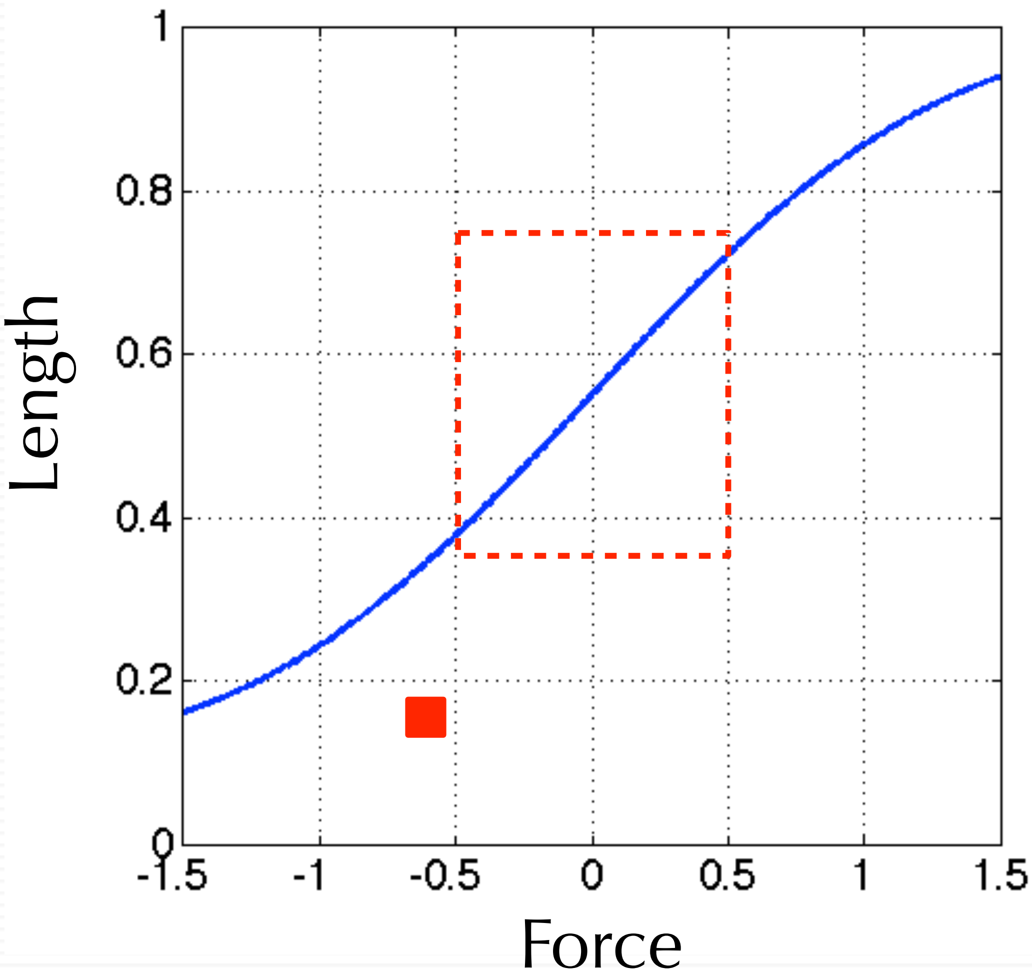
"If you pull a rubber band twice as hard, it will become twice as long."

"If you hit a ball twice as hard, it will go twice as far."

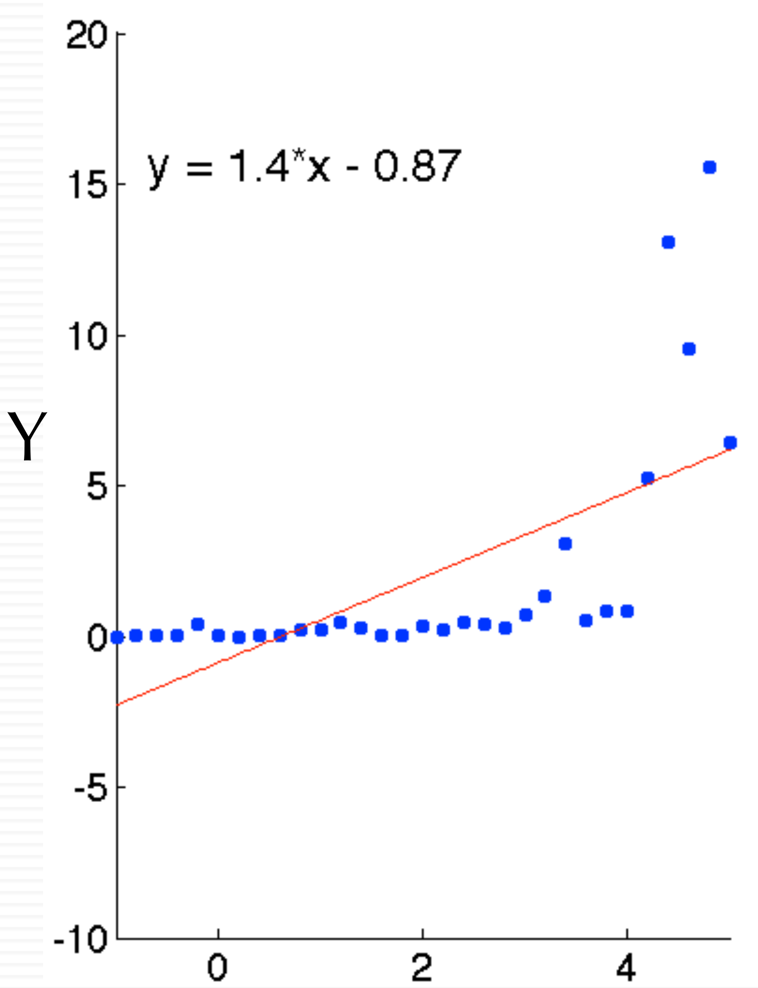
The Larmor Relation

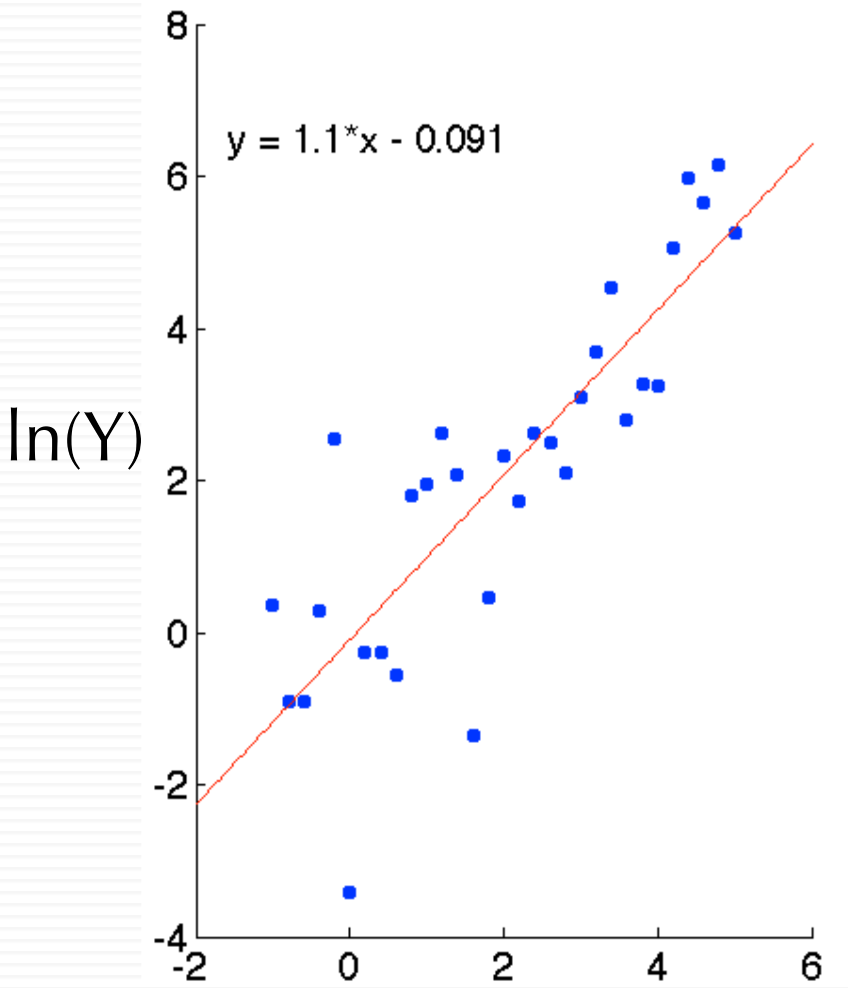


Linearity - Examples



Linearization





Moving Average

http://goldstocktrades.com/blog/wp-content/uploads/2010/09/Gld-9-20-10.jpg

2D Convolution

Reference Figure

		1		
	1	1	1	
1	1	1	1	1
	1	1	1	
		1		

Boxcar

2D Convolution

Reference Figure

	-1	
-1	5	-1
	-1	

Laplacian

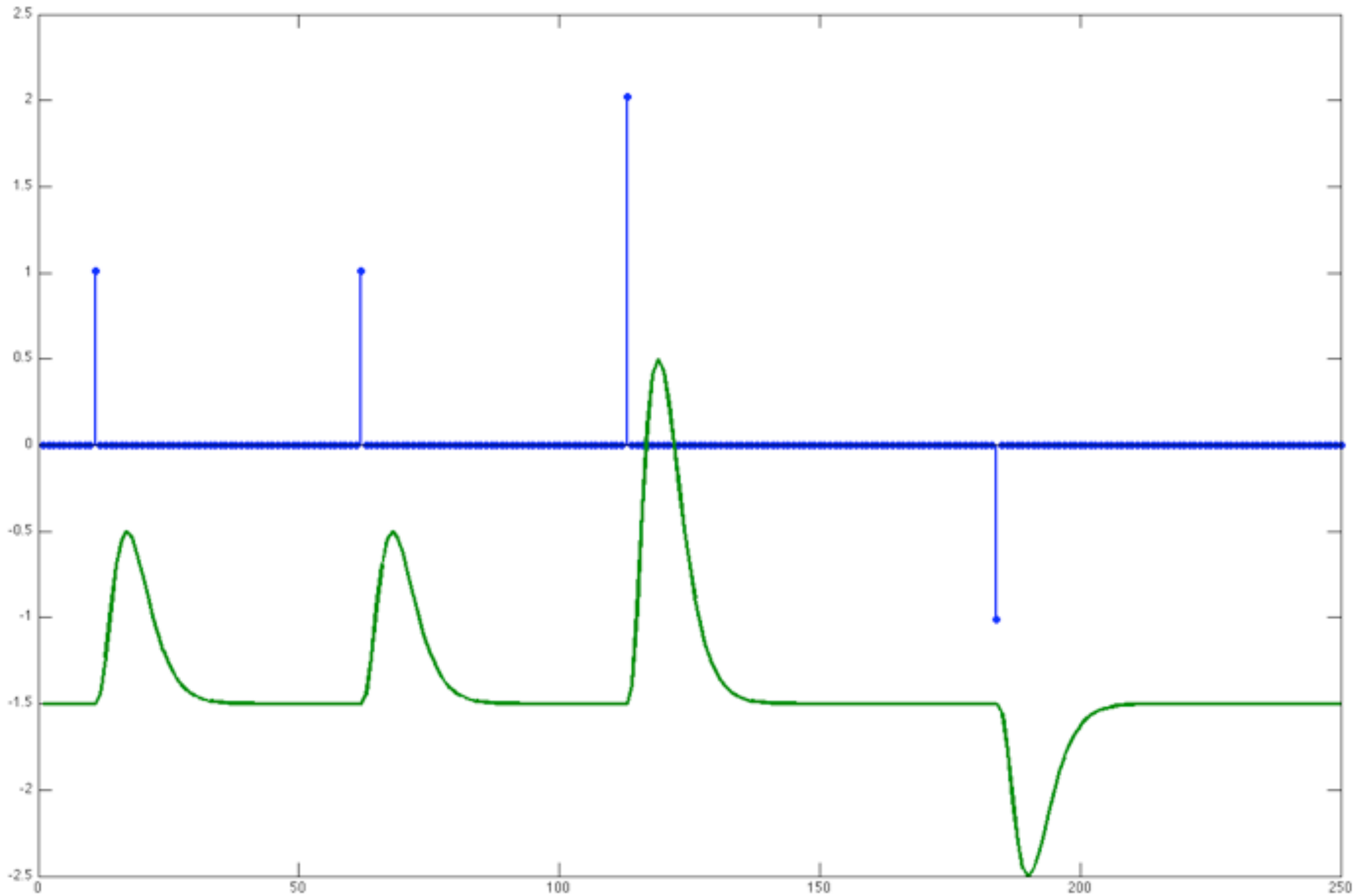
2D Convolution

Reference Figure

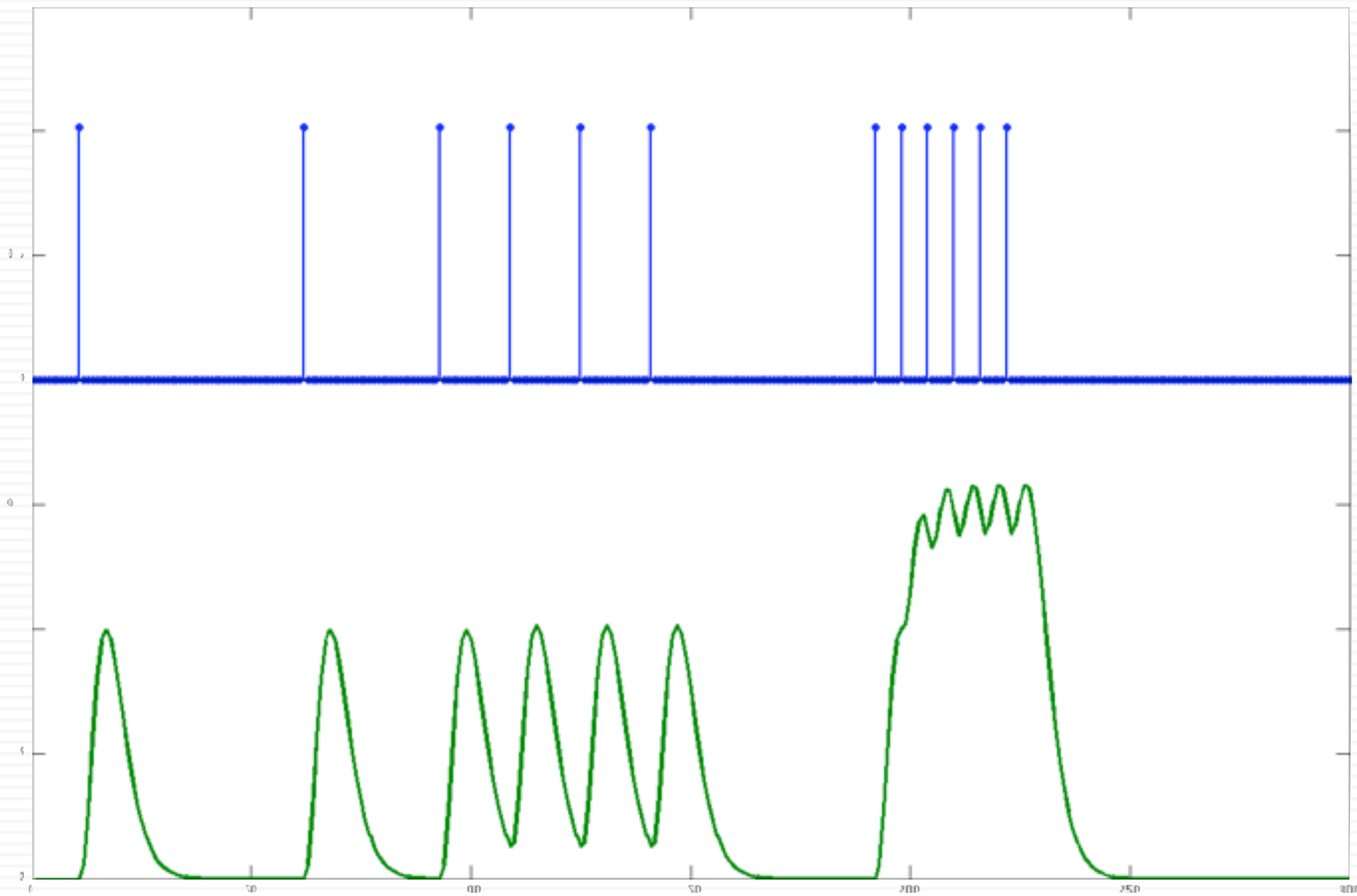
4	9	12	9	4
6	12	16	12	6
7	16	20	16	7
6	12	16	12	6
4	9	12	9	4

Gaussian

Impulse Response:
scale and shift

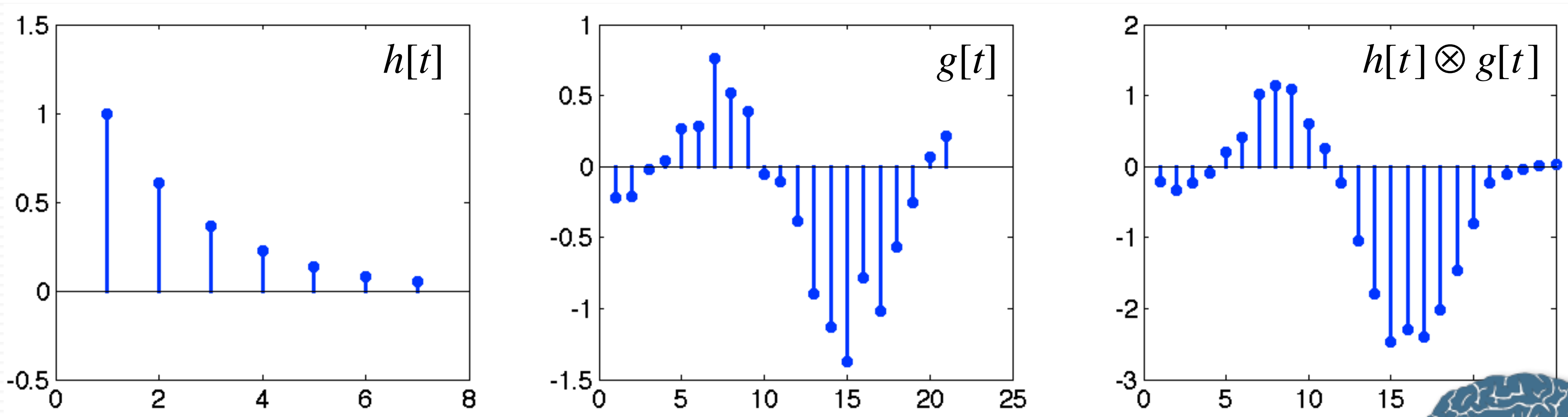


Impulse Response
superposition

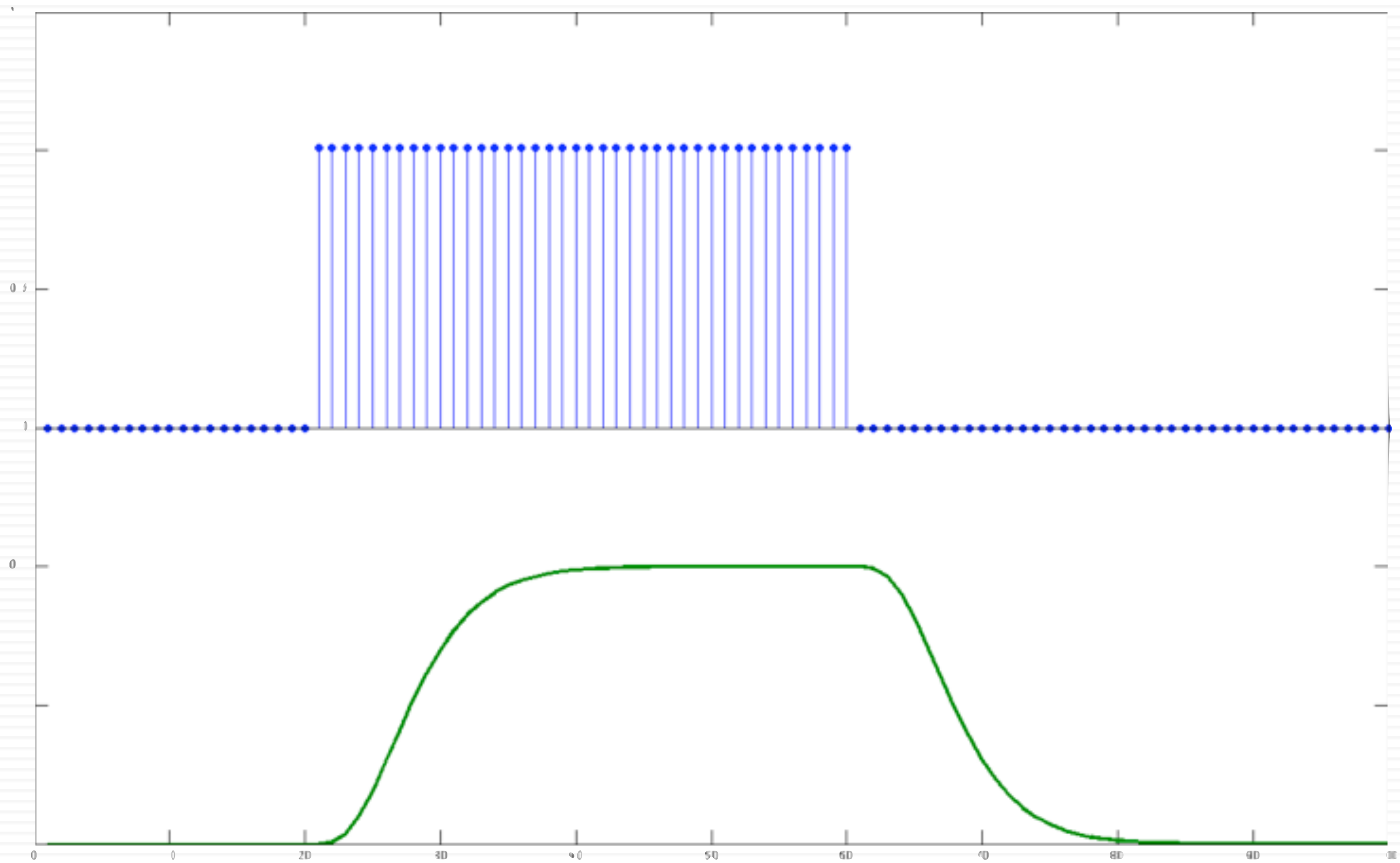


Impulse Responses

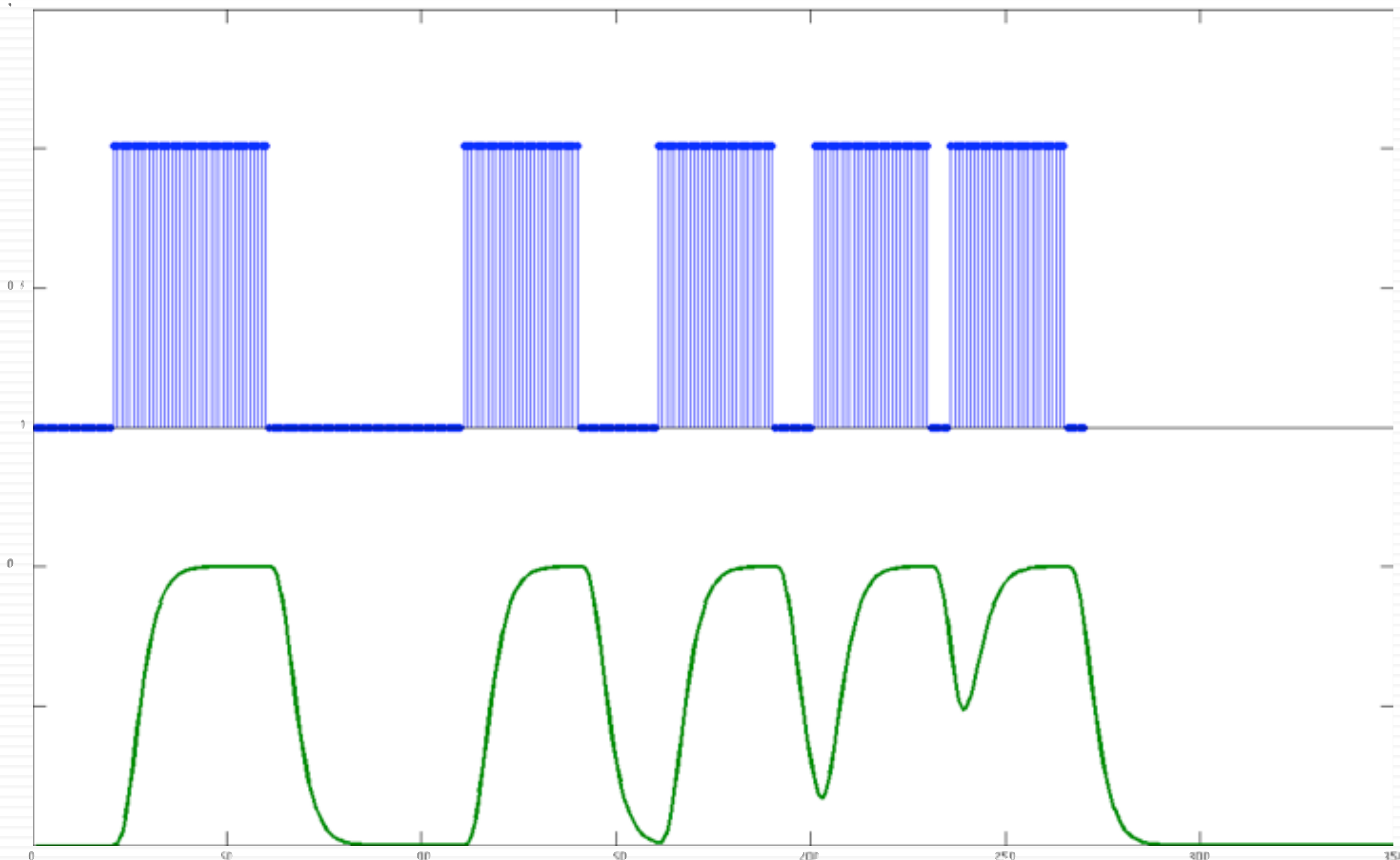
$$g[n] \otimes h[n] = \sum_{k=-\infty}^{\infty} g[k]h[n-k]$$



Rapid Events -> Blocks



Blocks and Baseline



Continuous Convolution

$$g[n] \otimes h[n] = \sum_{k=-\infty}^{\infty} g[k]h[n-k]$$

Discrete domain

$$g(t) \otimes h(t) = \int_{-\infty}^{\infty} g(\tau)h(t-\tau)d\tau$$

Continuous domain

See a great article convolution at:
http://en.wikipedia.org/wiki/Convolution

Properties of Convolution

Commutativity

$$f(t)\otimes g(t)=\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

$$u=t-\tau$$

$$=\int_{-\infty}^{\infty} f(t-u)g(u)d(t-u)$$

$$=\int_{-\infty}^{\infty} f(t-u)g(u)du$$

$$=\int_{-\infty}^{\infty} g(u)f(t-u)du$$

$$=g(t)\otimes f(t)$$

Properties of Convolution

Distributivity

$$f(t)\otimes(g(t)+h(t)) = \int_{-\infty}^{\infty} f(\tau)\left[g(t-\tau)+h(t-\tau)\right]d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau)g(t-\tau) + \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau$$

$$= f(t)\otimes g(t) + f(t)\otimes h(t)$$

Properties of Convolution

Associativity

$$f(t)\otimes(g(t)\otimes h(t)) = f(t)\otimes\int_{-\infty}^{\infty} g(\tau)h(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} f(\varphi)\otimes\int_{-\infty}^{\infty} g(\tau)h(t-\tau-\varphi)\,d\tau\,d\varphi$$

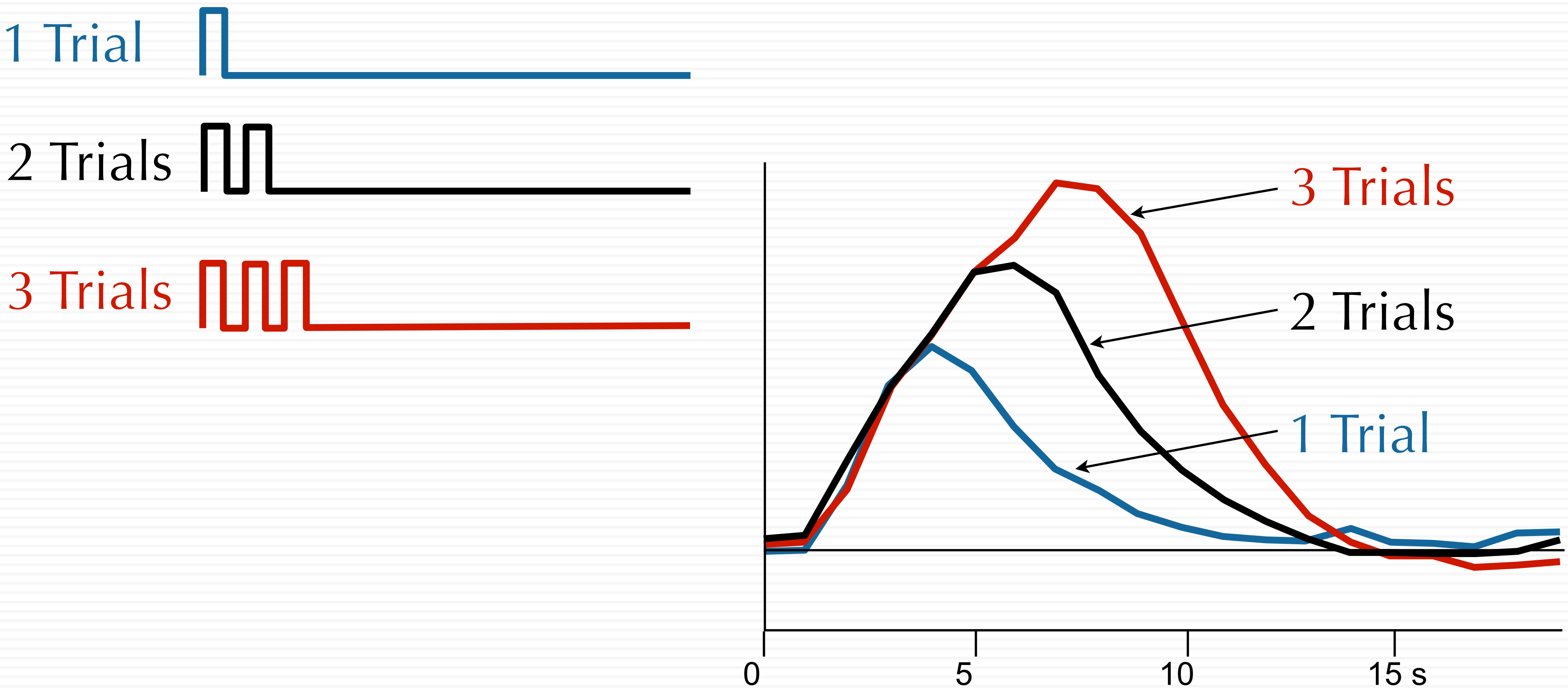
$$= \int_{-\infty}^{\infty} f(\varphi)\otimes\int_{-\infty}^{\infty} h(t-\tau-\varphi)\,d\varphi\big(g(\tau)\,d\tau\big)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big(f(\varphi)h(t-\tau-\varphi)\,d\varphi\big)\otimes g(\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} f(\varphi)h(t-\varphi)\,d\varphi\otimes g(\tau)$$

$$= \big(f(t)\otimes g(t)\big)\otimes h(t).$$

Time Invariance



Dale and Buckner 1997

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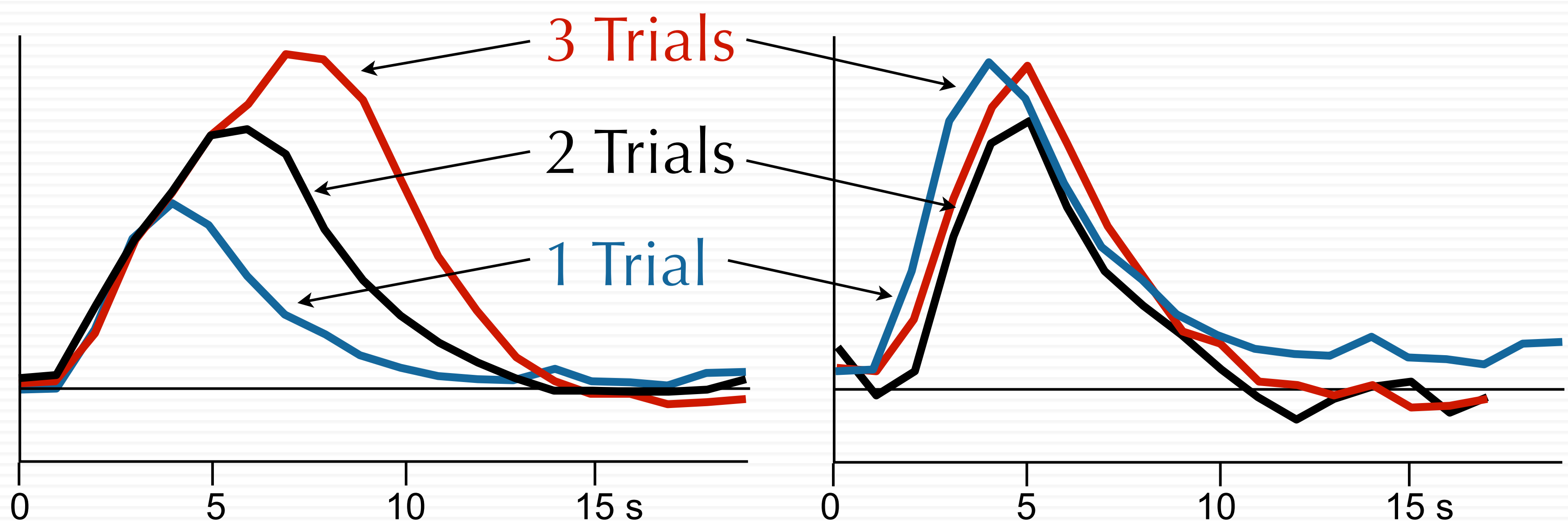
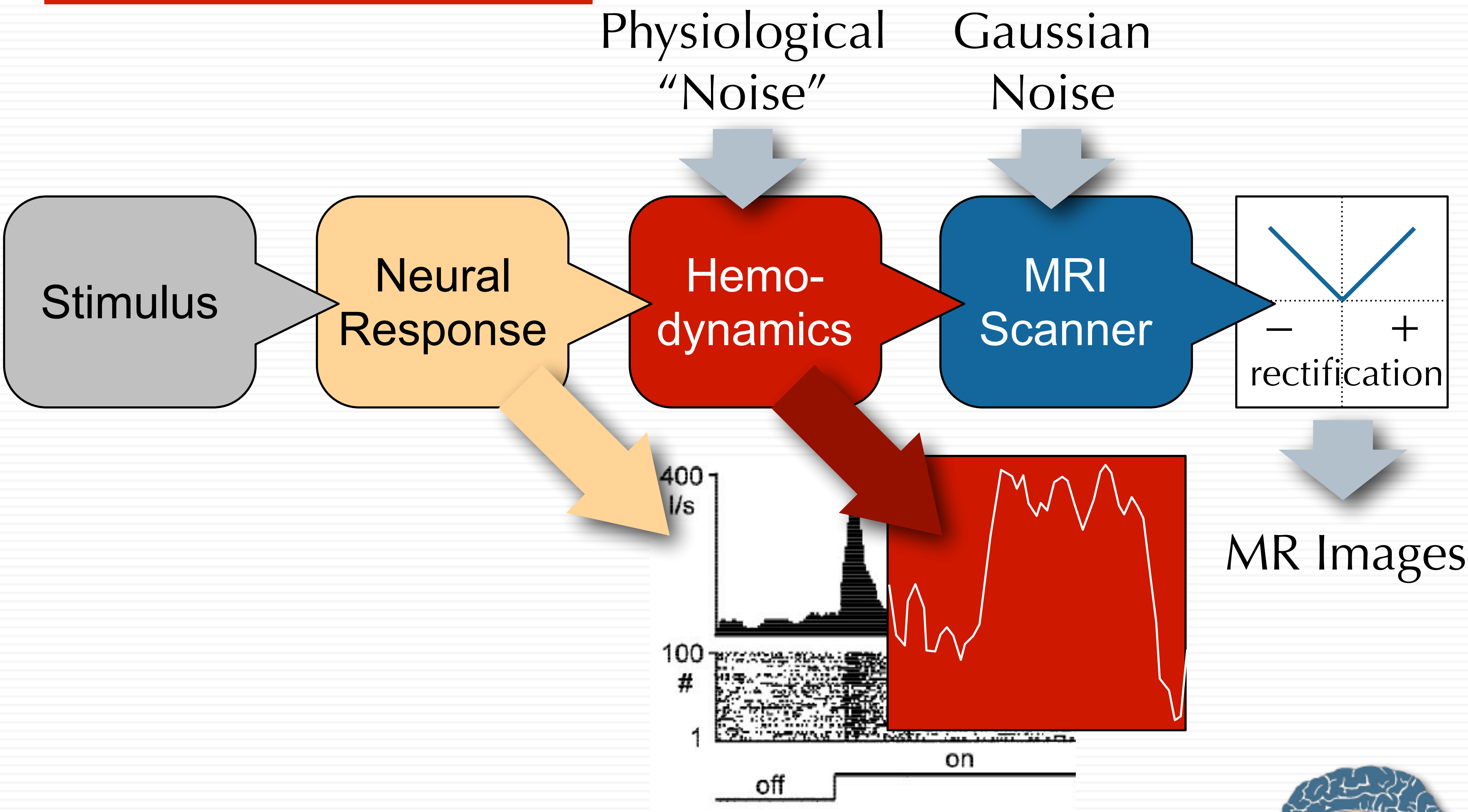
Time Invariance?

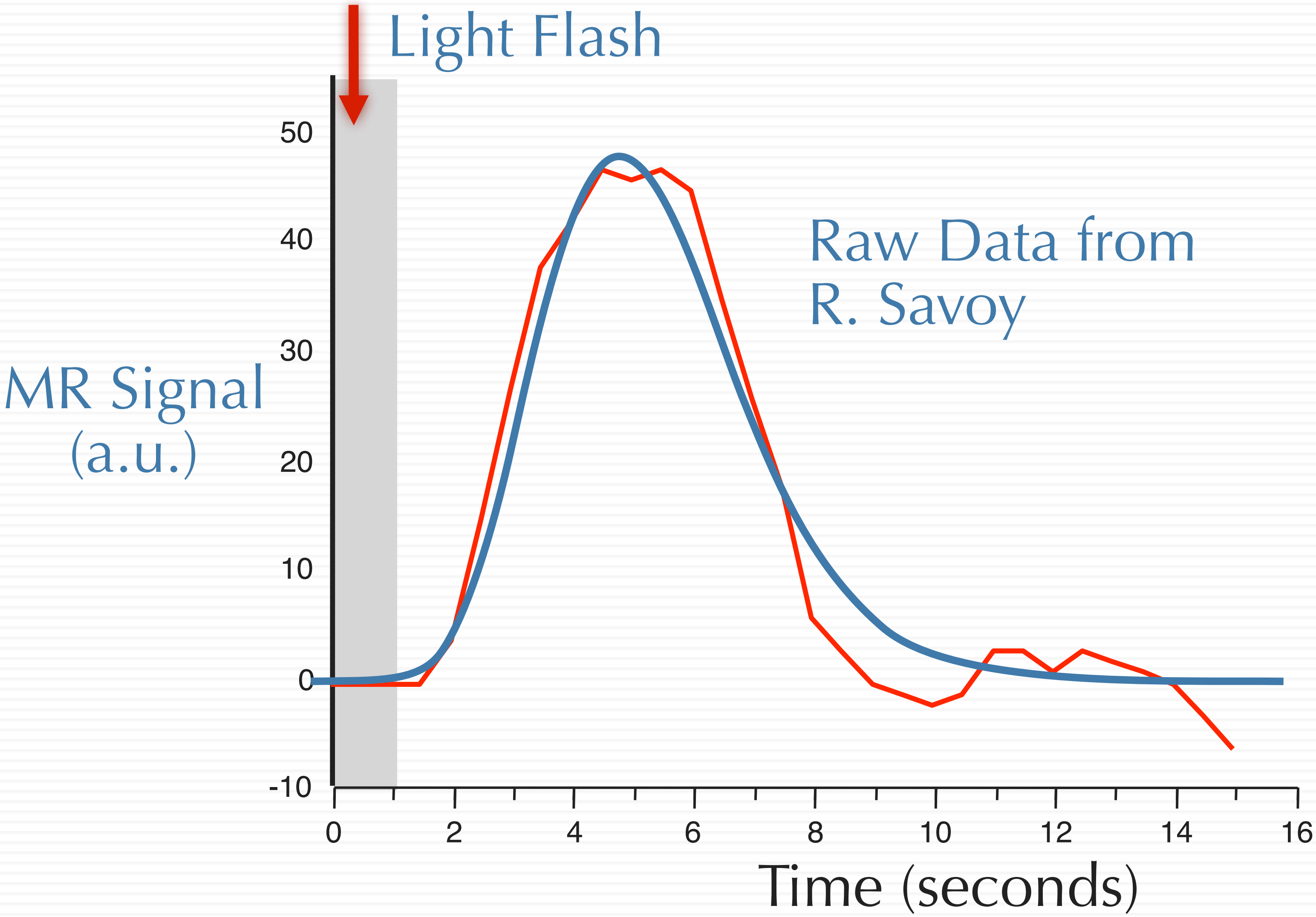


Dale and Buckner 1997

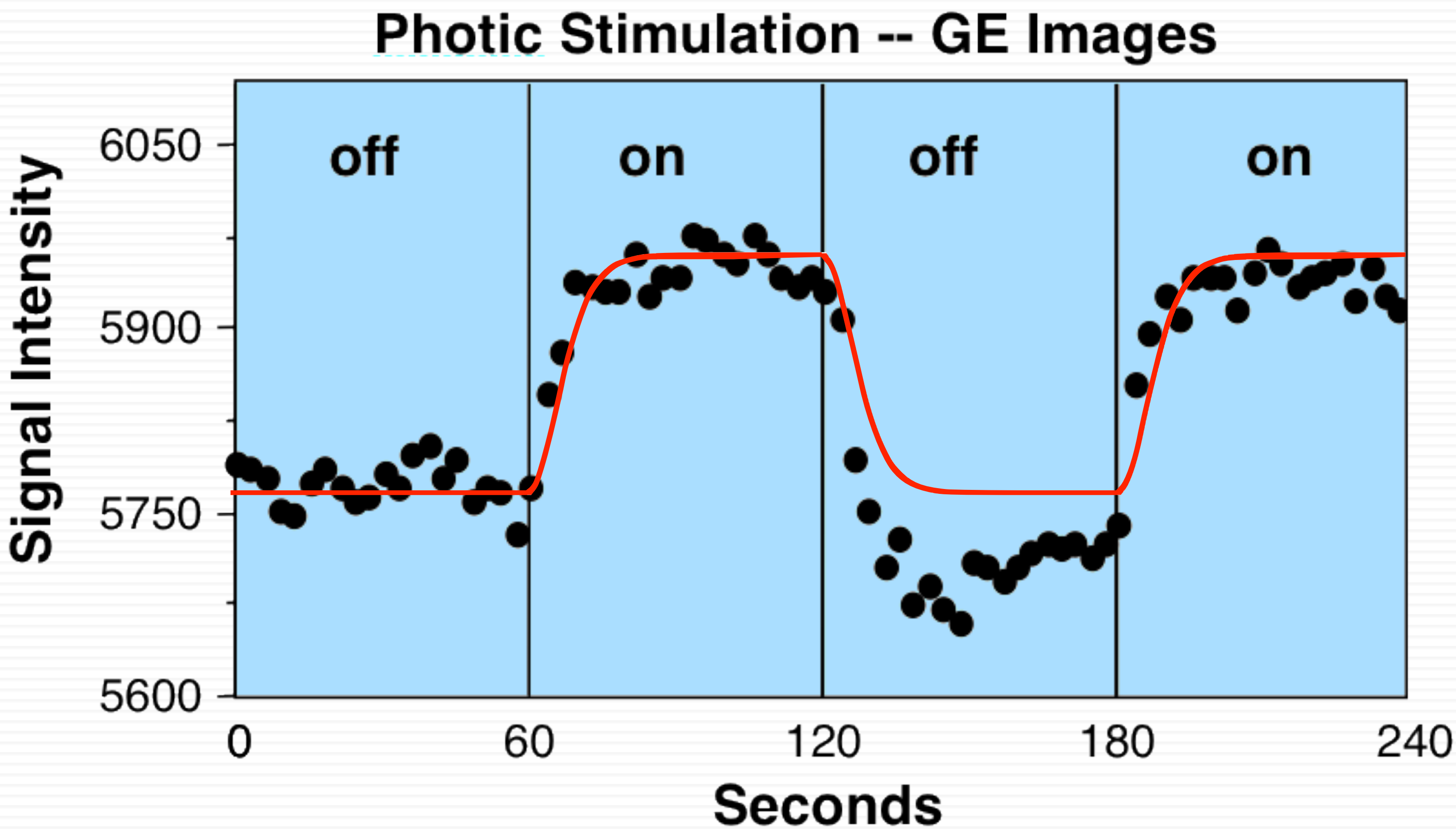
Transfer Function Model



Brain Impulse Response



Gradient-Recalled Echo



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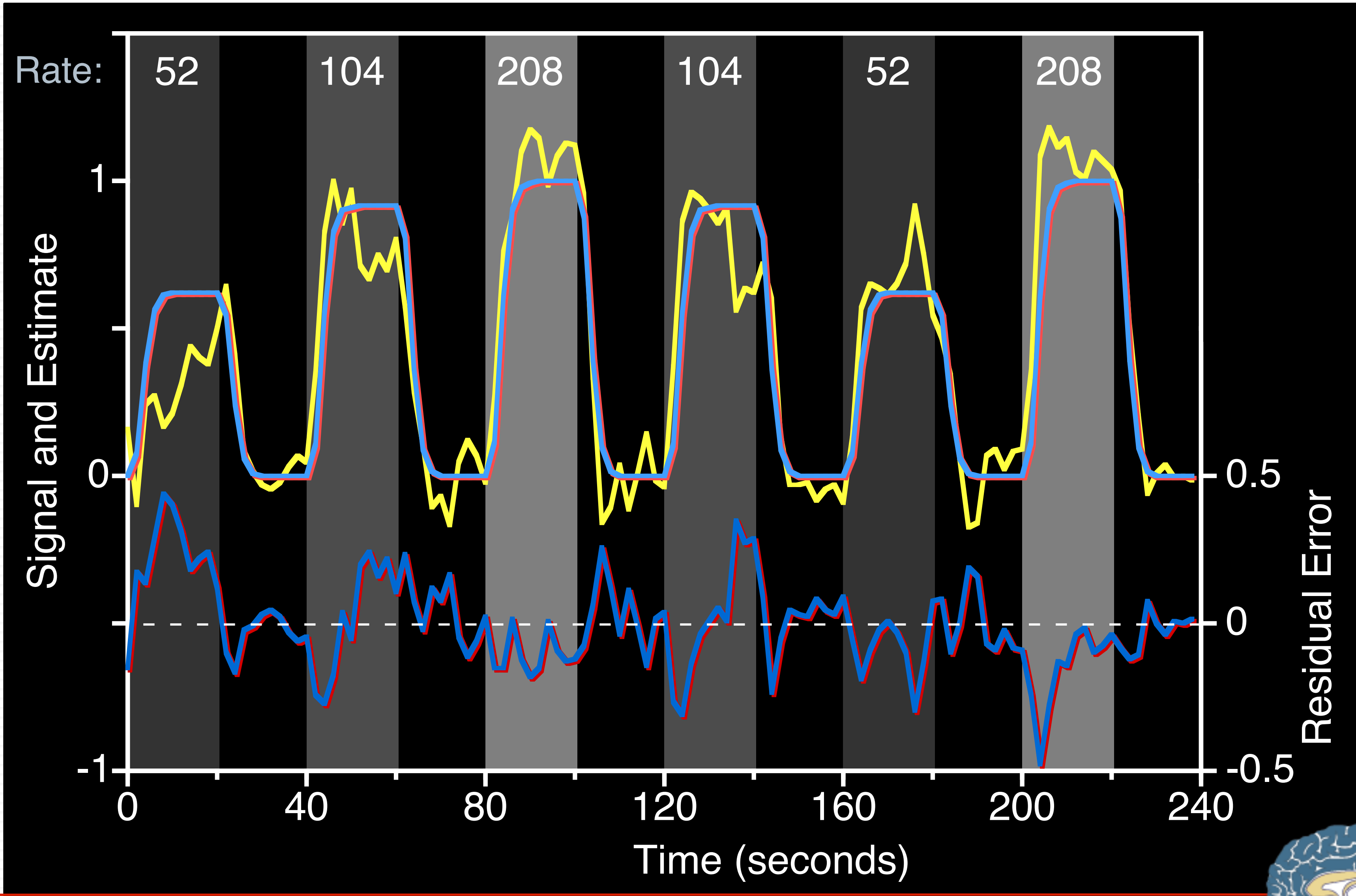
Amplitude-weighted Linear Estimate



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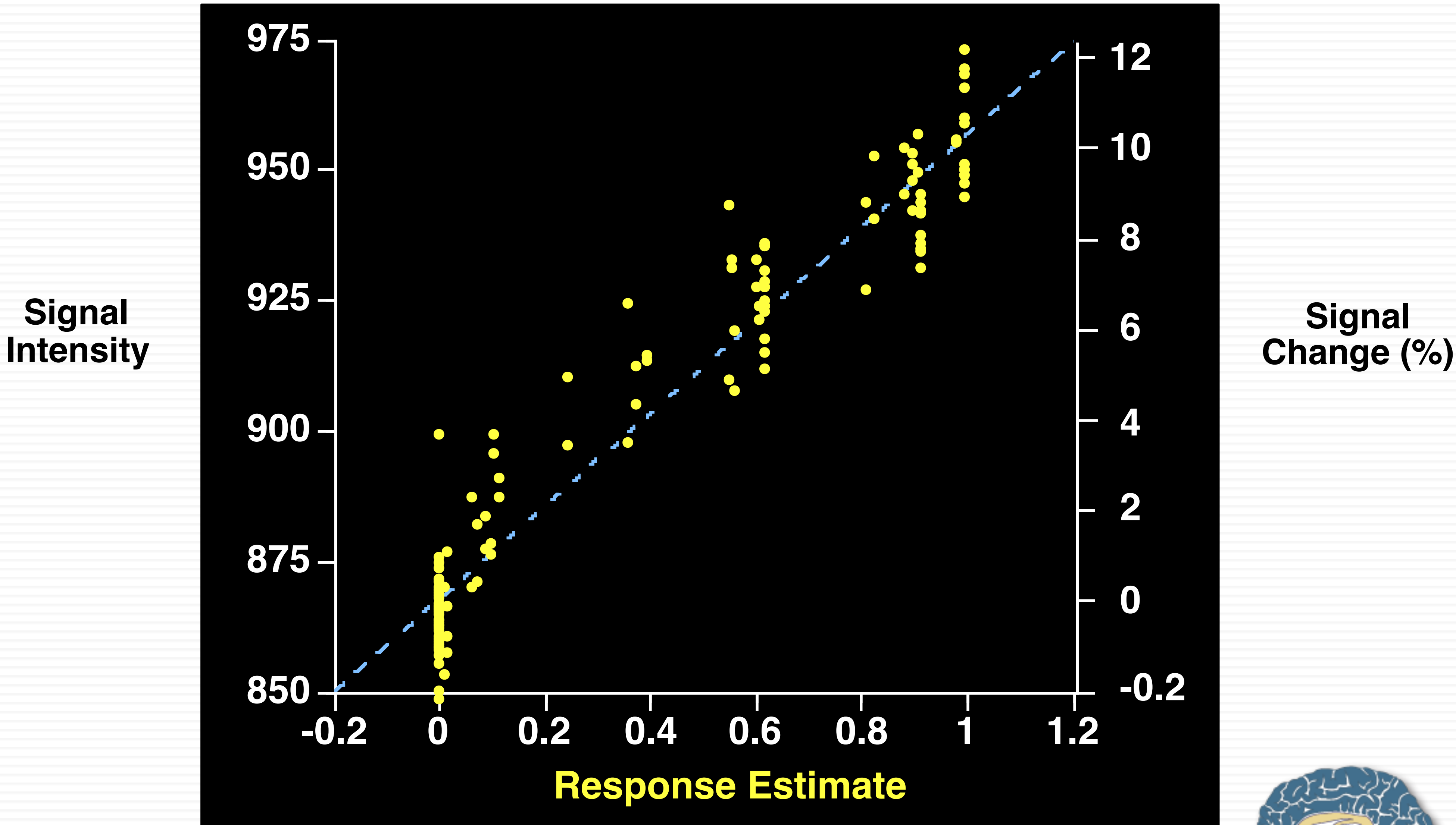
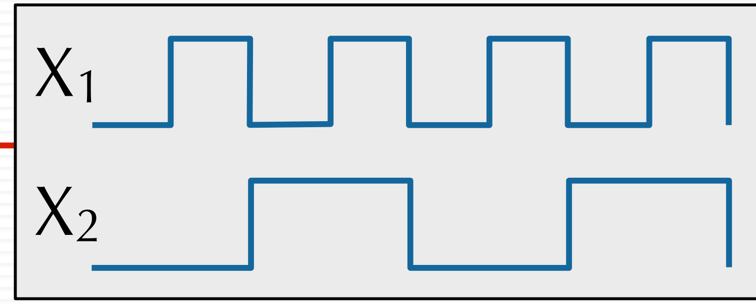
Estimated vs. Actual ƒMRI Response



General Linear Model

(matrix form)



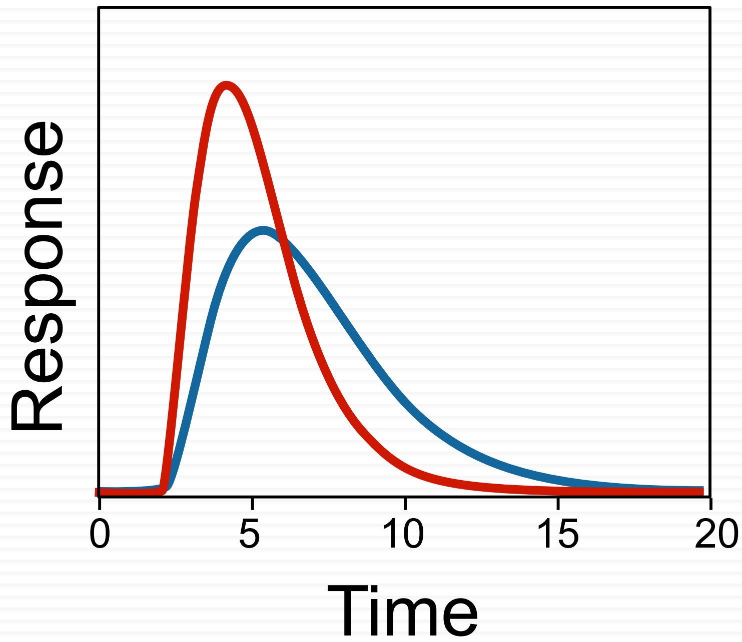
$$\mathbf{Y} = \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2 + \varepsilon$$

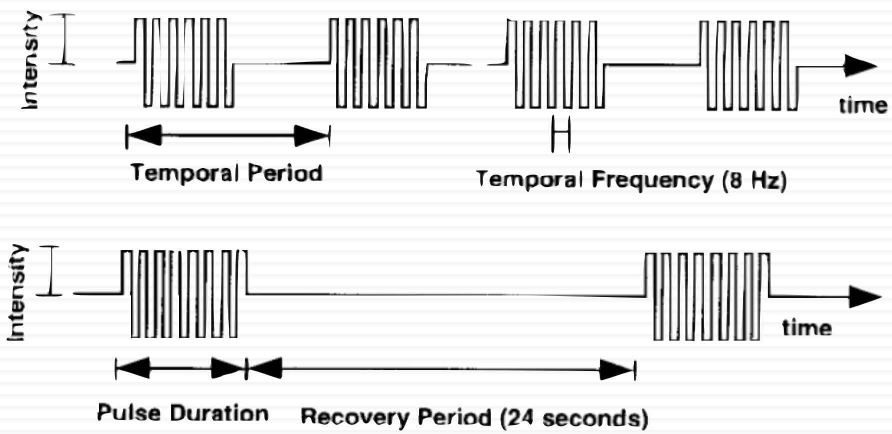
$$\mathbf{Y} = f(\mathbf{X}_1)\beta_1 + f(\mathbf{X}_2)\beta_2 + \varepsilon$$

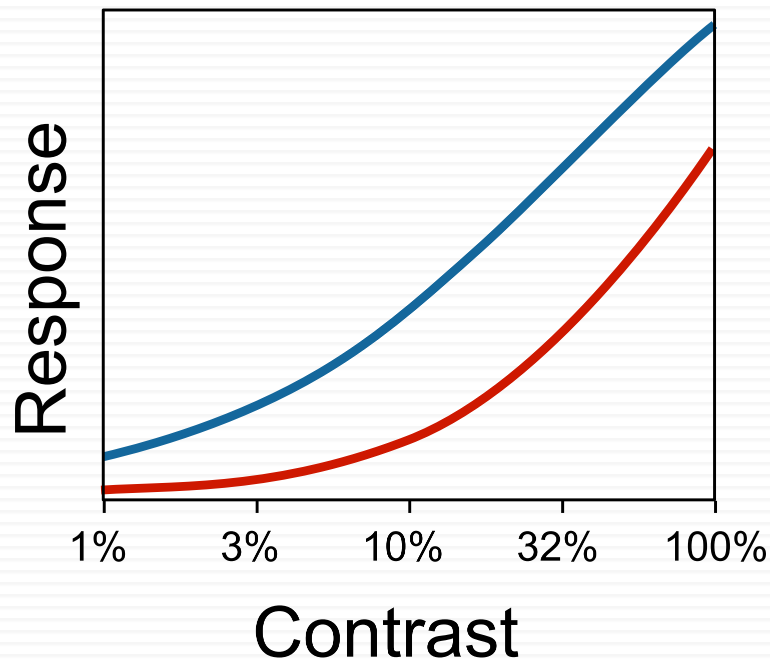
$$\mathbf{Y} = \ln(\mathbf{X}_1)\beta_1 + \ln(\mathbf{X}_2)\beta_2 + \varepsilon$$

$$\mathbf{Y} = \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2 + \mathbf{X}_1\mathbf{X}_2\beta_3 + \varepsilon$$

Time Invariance (redux)







Boynton, et al., 1996

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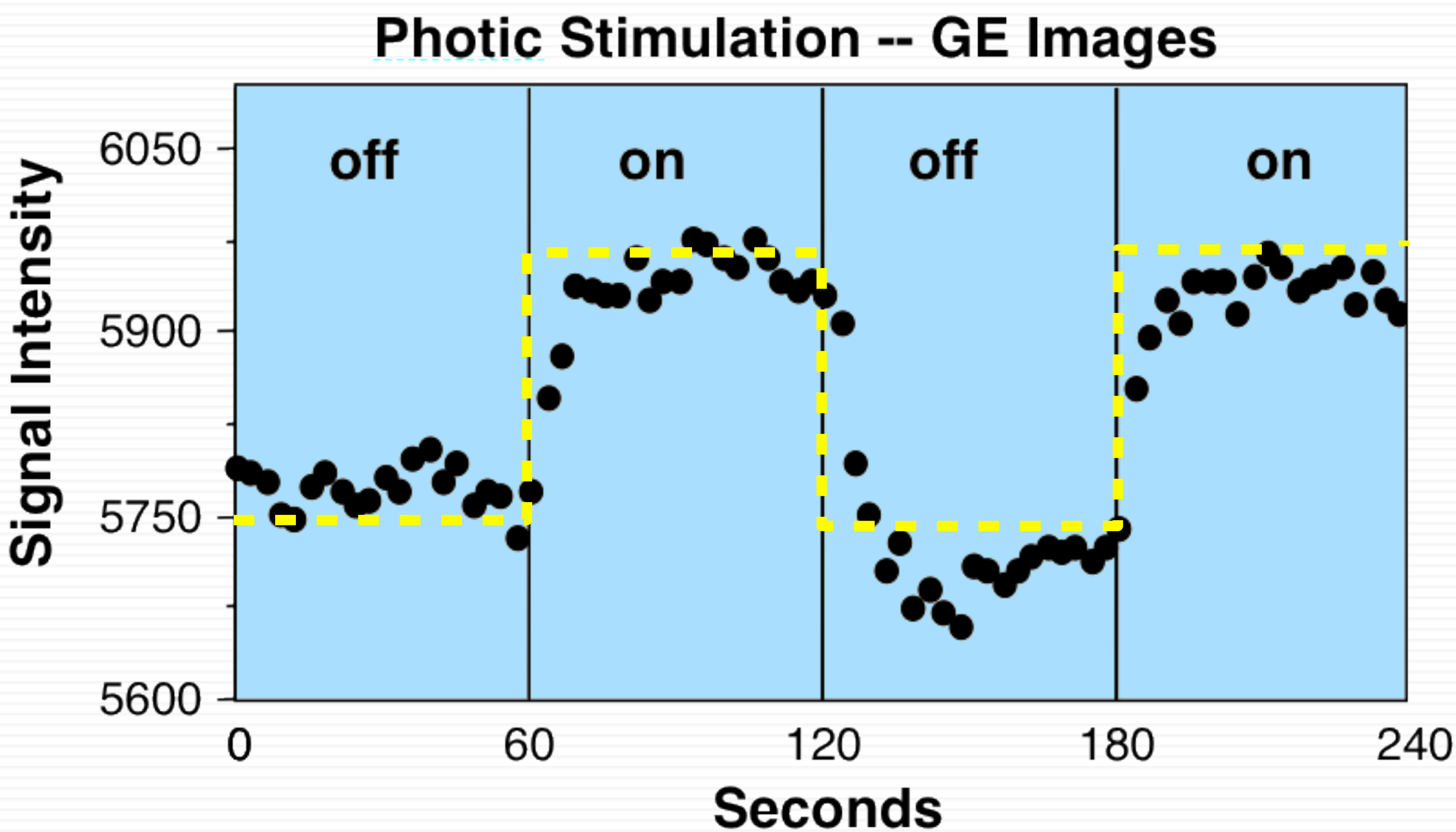
Gradient-Recalled Echo



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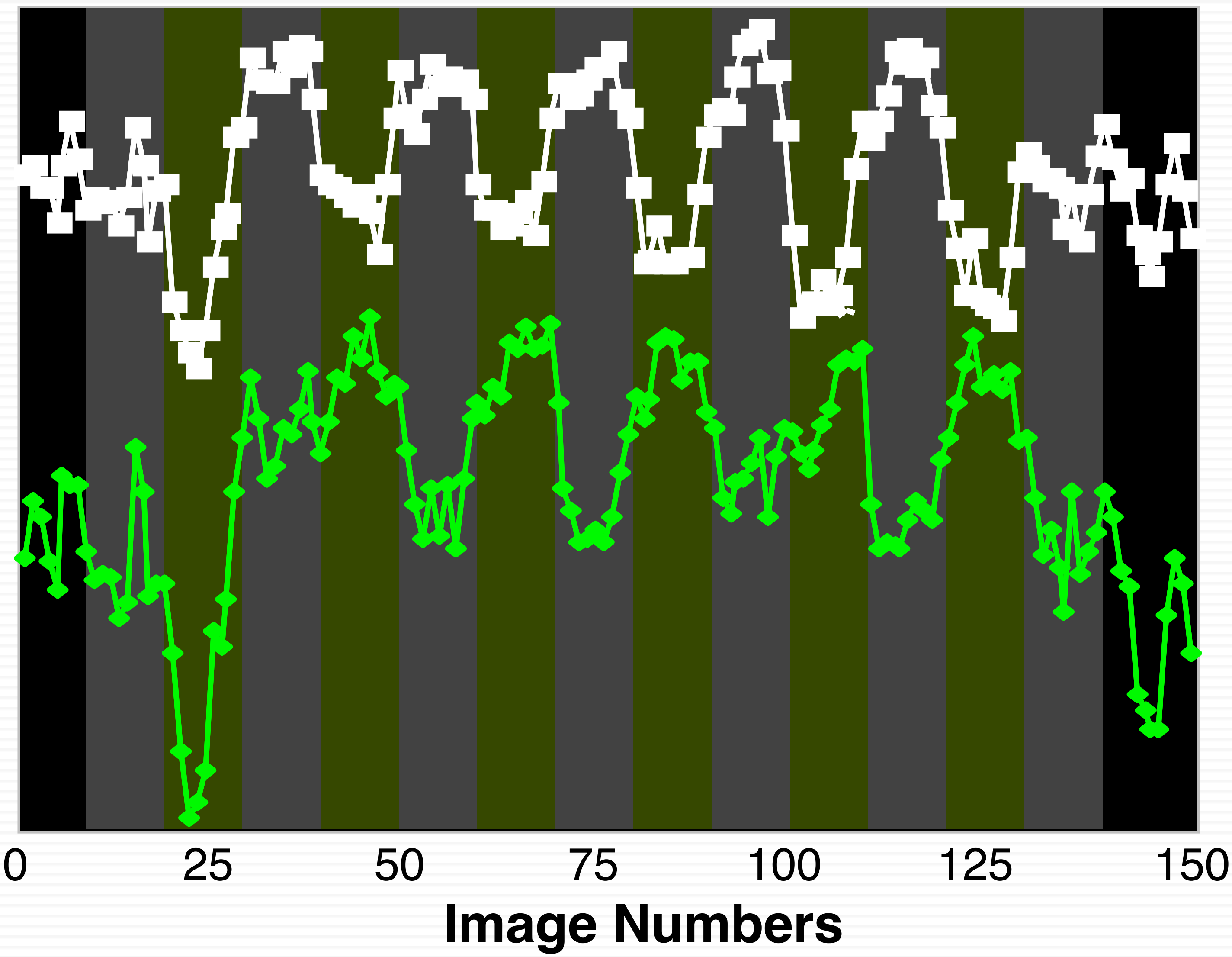
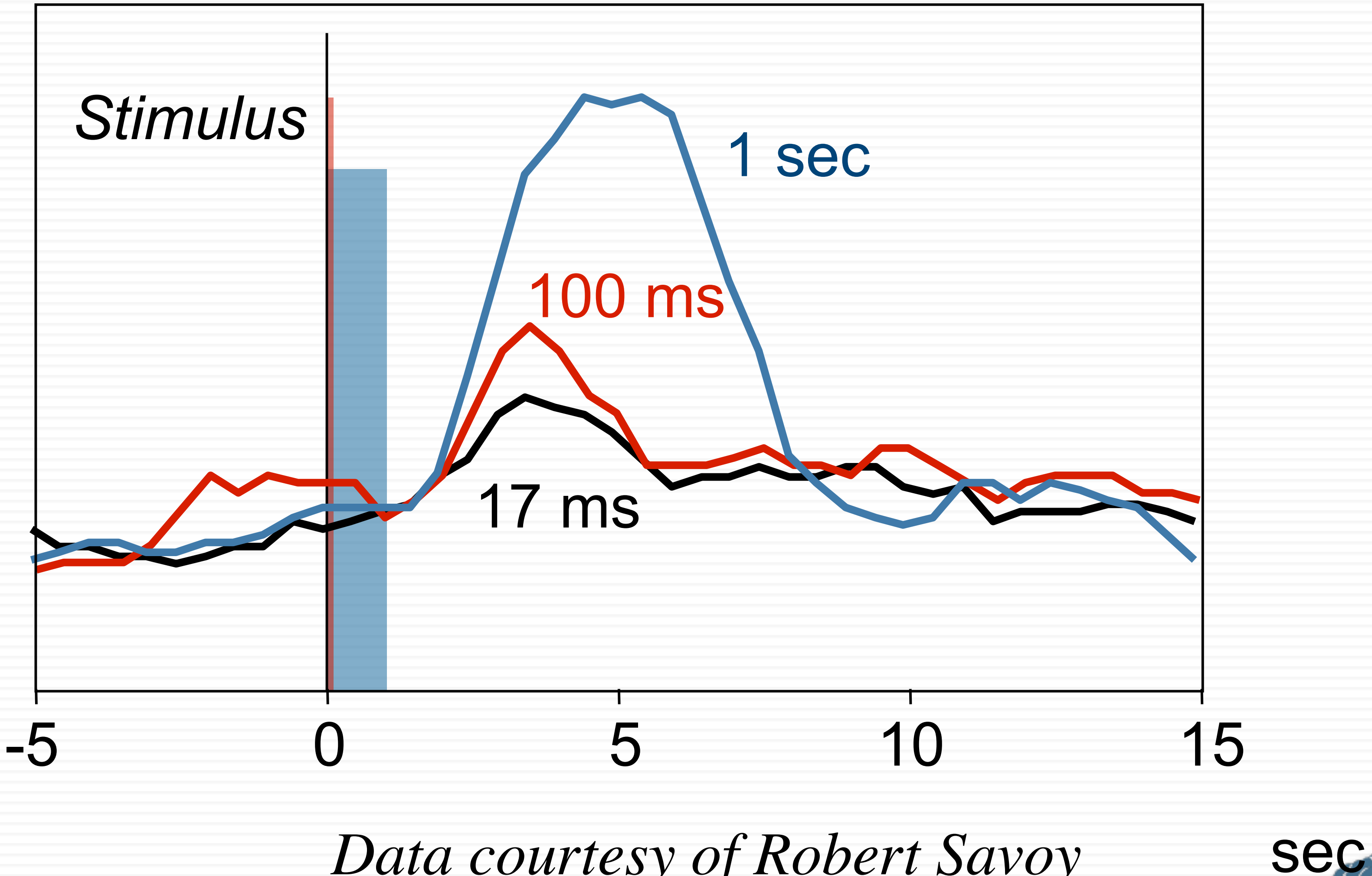
Hemifield Alternation 20 seconds

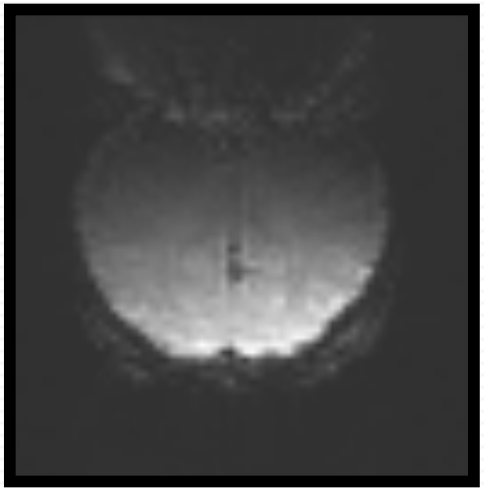


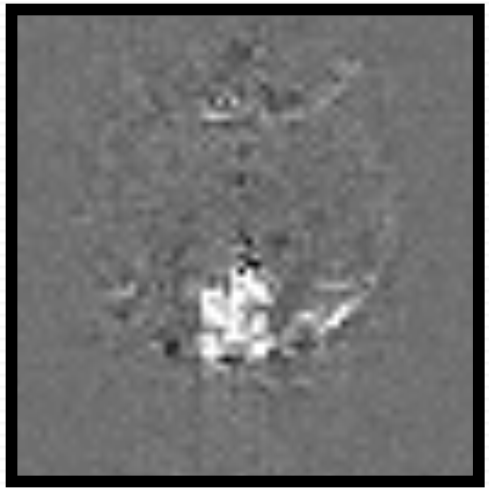
Response Latency vs. Stimulus Duration
Average of 10 recordings

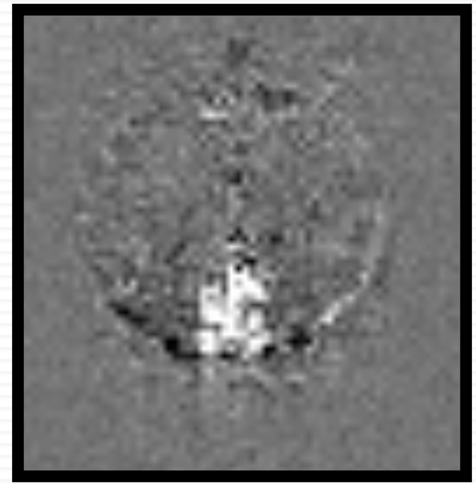


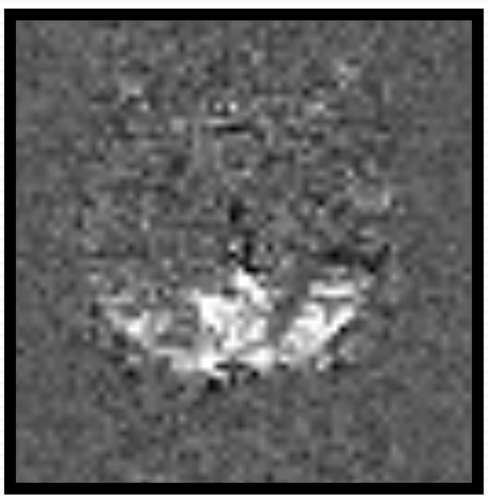
Data courtesy of Robert Savoy

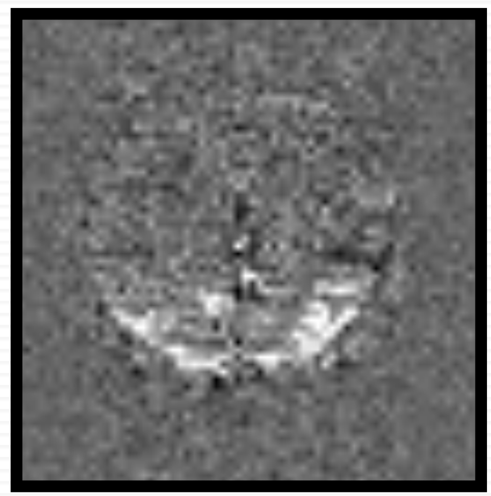
Binocular vs Monocular Activation


baseline


Binocular


Monocular


Bino minus Mono


Extrastriate activation

The Euler Relation

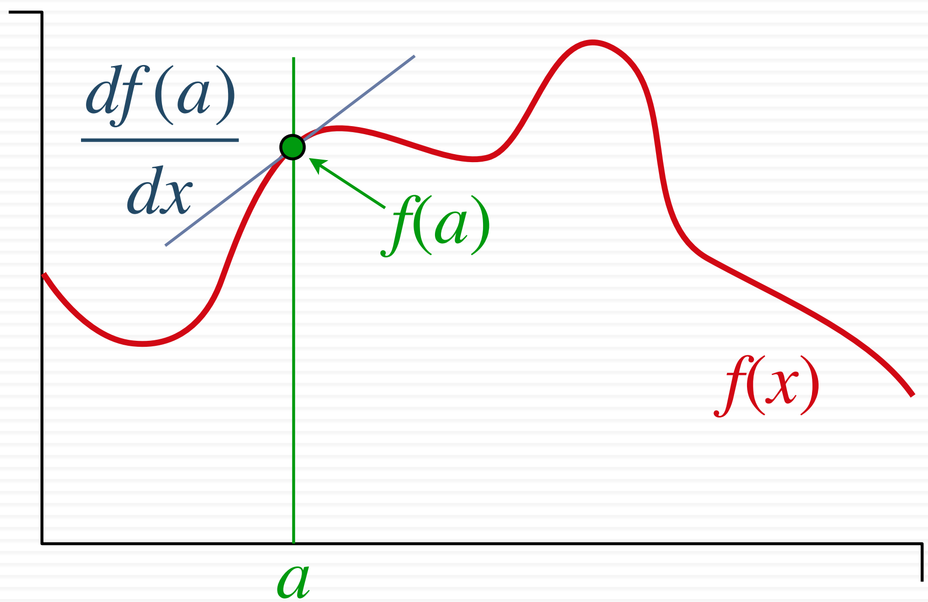
$$e^{ix} = \cos x + i \sin x$$

The Taylor Series

$$f(x) = f(a)(x-a)^0 + \frac{df(a)}{dx}\frac{(x-a)^1}{1!} + \frac{d^2 f(a)}{dx^2}\frac{(x-a)^2}{2!} + \ldots + \frac{d^n f(a)}{dx^n}\frac{(x-a)^n}{n!} + \ldots$$

$$= \sum_0^\infty f^{(n)} \frac{(x-a)^n}{n!}$$

$$\approx f(a) + (x-a)\frac{df}{dx}$$



The Euler Relation - getting there

Taylor Series

$$f(x) = f(a)(x-a)^0 + \frac{df(a)}{dx}\frac{(x-a)^1}{1!} + \frac{d^2 f(a)}{dx^2}\frac{(x-a)^2}{2!} + \ldots + \frac{d^n f(a)}{dx^n}\frac{(x-a)^n}{n!} + \ldots$$

$$= \sum_0^{\infty} f^{(n)}(a)\frac{(x-a)^n}{n!}$$

McLaurin Series

$$f(x) = f(0) + \frac{df(0)x}{dx} + \frac{d^2 f(0)}{dx^2}\frac{x^2}{2!} + \frac{d^3 f(0)}{dx^3}\frac{x^3}{3!} + \ldots + \frac{d^n f(0)}{dx^n}\frac{x^n}{n!} + \ldots$$

$$f(x) = \sum_{n=1}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$$

The Euler Relation - getting there

$$f(x) = \sum_{n=1}^{\infty} f^{(n)} \frac{x^n}{n!} + f(0)$$

$$\sin(x) = \sin(0) + x\cos(0) - \frac{x^2 \sin(0)}{2!} - \frac{x^3 \cos(0)}{3!} + \frac{x^4 \sin(0)}{4!} + \frac{x^5 \cos(0)}{5!} + \ldots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\cos(x) = \cos(0) + x\sin(0) - \frac{x^2 \cos(0)}{2!} - \frac{x^3 \sin(0)}{3!} + \frac{x^4 \cos(0)}{4!} + \frac{x^5 \sin(0)}{5!} + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!} + \frac{u^6}{6!} + \frac{u^7}{7!} + \ldots$$

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The Euler Relation - getting there

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!} + \frac{u^6}{6!} + \frac{u^7}{7!} + \ldots$$

$$\boxed{e^{ix}} = 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \frac{i^6x^6}{6!} + \frac{i^7x^7}{7!} + \ldots$$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + i\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right]$$

$$\boxed{= \cos(x) + i\sin(x).}$$

Fourier transform

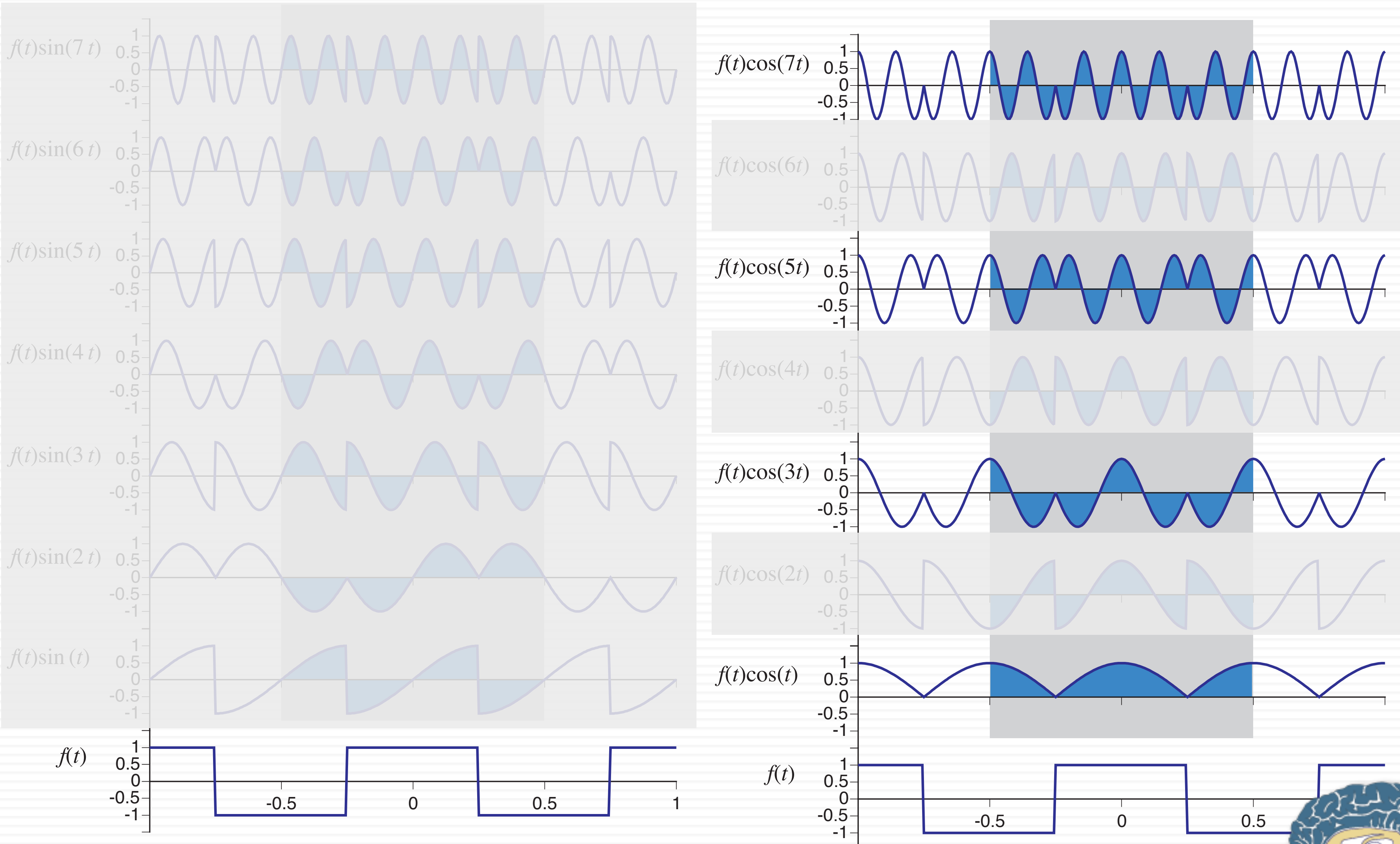
$$\mathscr{F}(s) = \mathscr{F}\big(f(x)\big)$$
$$= \int_{-\infty}^{\infty} f(x)\left[\cos(2\pi sx) - i\sin(2\pi sx)\right]dx$$
$$= \int_{-\infty}^{\infty} f(x)e^{-2\pi isx}\,dx$$

$$\mathscr{F}^{-1}\big(\mathscr{F}(s)\big) = f(x)$$
$$= \int_{-\infty}^{\infty} f(x)e^{+2\pi isx}\,dx.$$

Let $x = \dfrac{\omega}{2\pi}$:

$$\mathscr{F}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$$
$$\mathscr{F}^{-1}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)e^{2\pi i\omega x}\,d\omega.$$

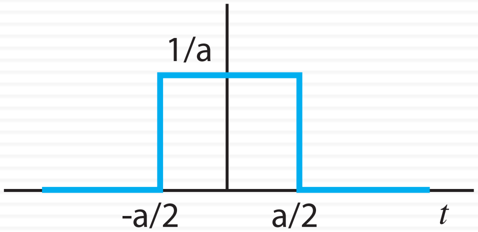
Fourier transform



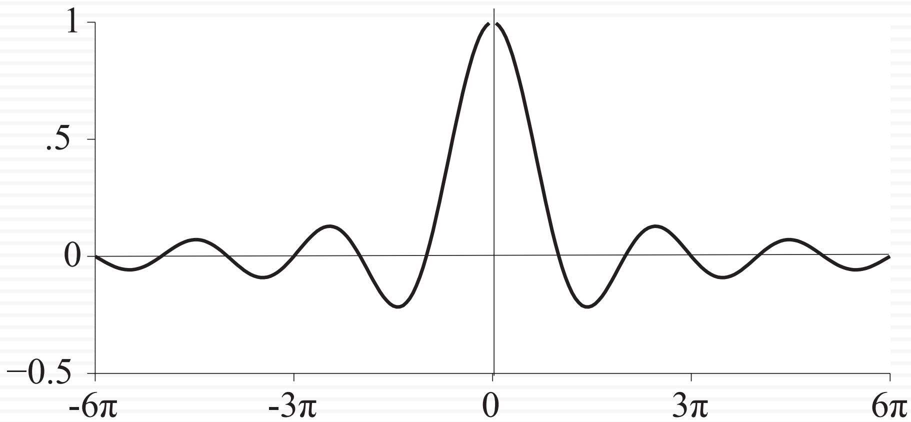
Using the Fourier Transform

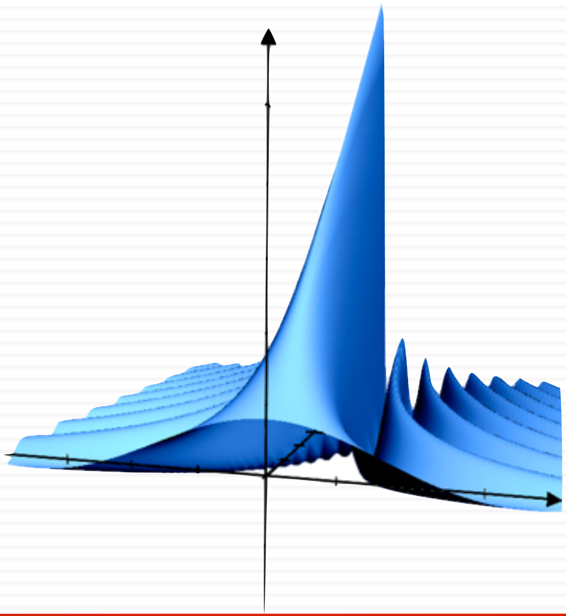
- Important Pairs - Square and sinc:

$$f(t)=\begin{cases} \frac{1}{a} \text{ for } -\frac{a}{2}<t<\frac{a}{2} \\ 0 \text{ otherwise} \end{cases}$$



$$f(s)=\mathscr{F}(f(t))=\frac{\sin(\pi as)}{\pi as}$$





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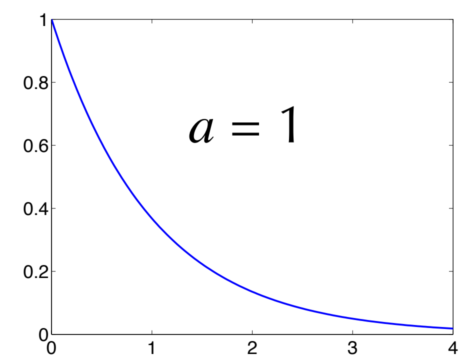
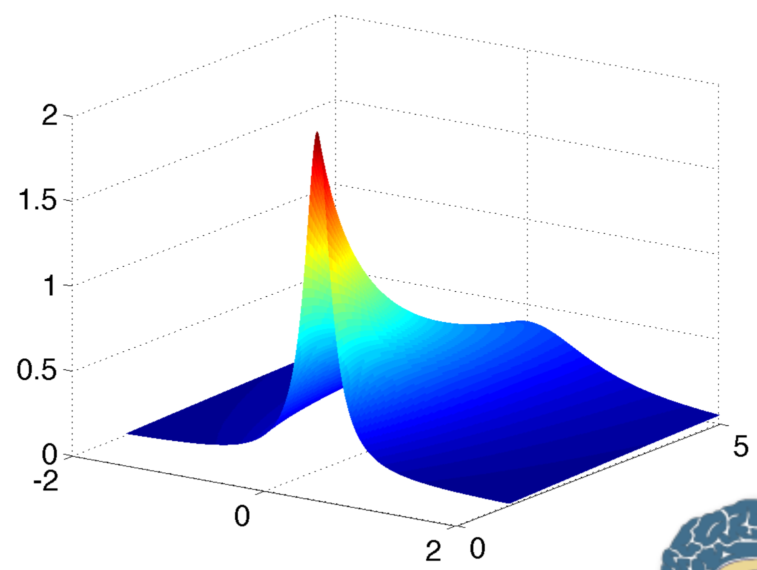
Using the Fourier Transform

- Important Pairs - Exponential and Lorenztian:

$$f(t) = e^{-a|x|}$$



$$\mathscr{F}(s) = \frac{2a}{4\pi^2 s^2 + a^2}$$

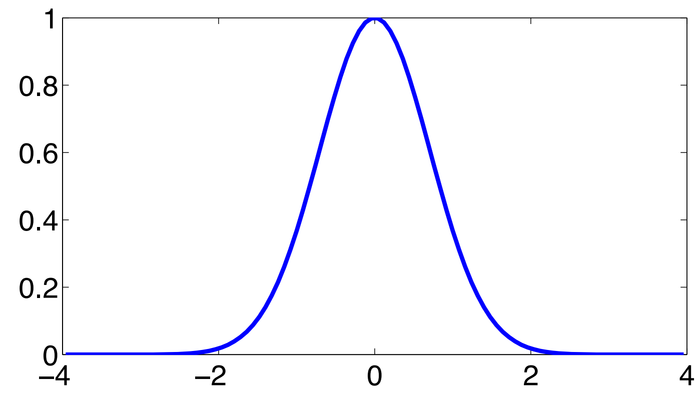


Using the Fourier Transform

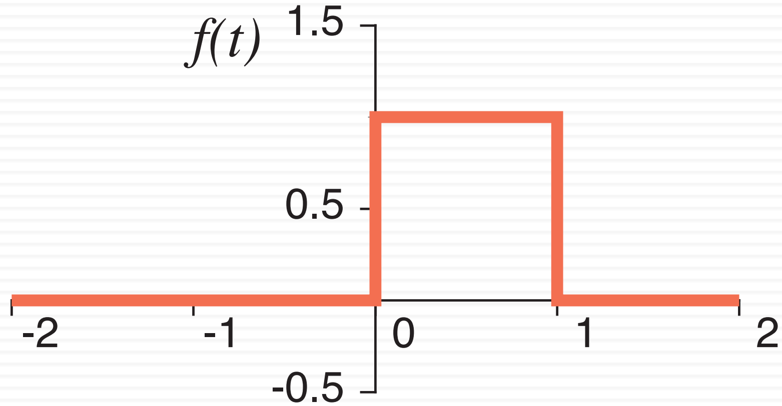
- Important Pairs - Gaussian:

$$f(t) = e^{-ax^2}$$

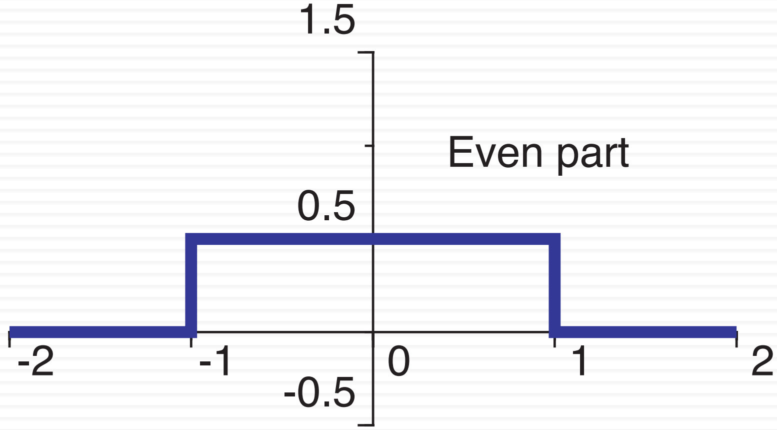
$$\mathscr{F}(s) = \frac{\sqrt{\pi} e^{-\pi x^2}}{\sqrt{a}}.$$



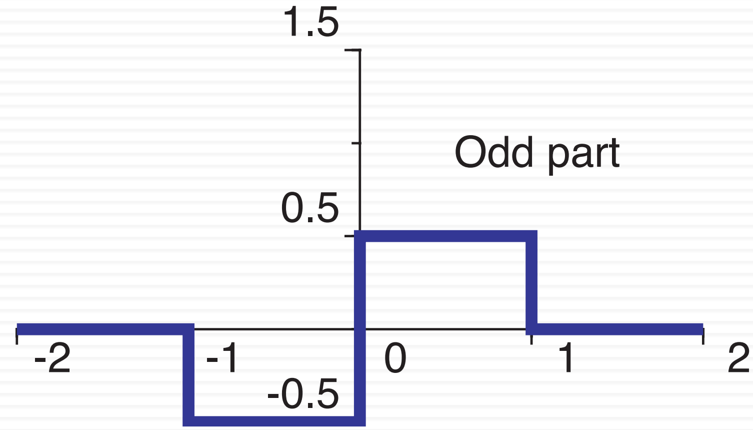
Odd and Even



f(t)
1.5
0.5
-0.5
-2
-1
0
1
2



Even part
1.5
0.5
-0.5
-2
-1
0
1
2



Odd part
1.5
0.5
-0.5
-2
-1
0
1
2

Odd and Even

if a function is:	*its Fourier transform is:*
Real and Even	Real and Even
Real and Odd	Imaginary and Odd
Imaginary and Even	Imaginary and Even
Complex* and Even	Complex and Even
Complex and Odd	Complex and Odd
Real and Asymmetrical	Complex and Hermitian
Imaginary and Asymmetrical	Complex and anti-Hermitian
Real Even + Imaginary Odd	Real
Real Odd + Imaginary Even	Imaginary
Even	Even
Odd	Odd

Shift Theorem

$$\mathscr{F}[f(t-a)](s) = \int_{-\infty}^{\infty} f(t-a)e^{-2\pi ist}dt$$

$$= e^{-2\pi isa}\int_{-\infty}^{\infty} f(t-a)e^{-2\pi ist}e^{2\pi isa}dt$$

$$= e^{-2\pi isa}\int_{-\infty}^{\infty} f(t-a)e^{-2\pi is(t-a)}dt.$$

Let $u = t-a$ and $du = dt$:

$$= e^{-2\pi isa}\int_{-\infty}^{\infty} f(u)e^{-2\pi isu}du$$

$$= e^{-2\pi isa}\mathscr{F}[f(t)](s)$$

$$= (\cos(2\pi sa) - i\sin(2\pi sa))\mathscr{F}[f(t)](s)$$

Convolution Theorem

$$\mathcal{F}(f(x) \otimes g(x)) = \mathcal{F}(f(x))\mathcal{F}(g(x))$$

$$\mathcal{F}(f(x)g(x)) = \mathcal{F}(f(x)) \otimes \mathcal{F}(g(x)).$$