Linearity / Time Invariance

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Linearity

- A system is *linear* if and only if:

$$f(Ax) = Af(x)$$

- It is time time invariant if

$$y(t) = f(x(t)) \text{ and}$$
$$y(t+\delta) = f(x(t+\delta))$$

for all δ.

Linear Systems Approach

In an LTI system, given two inputs A & B:

$$f(A + B) = f(A) + f(B)$$

$x(t)$ → [Linear / Time-Invariant / (LTI) System] → $y(t) = f[x(t)]$

- Linear
- Time-Invariant
- (LTI) System

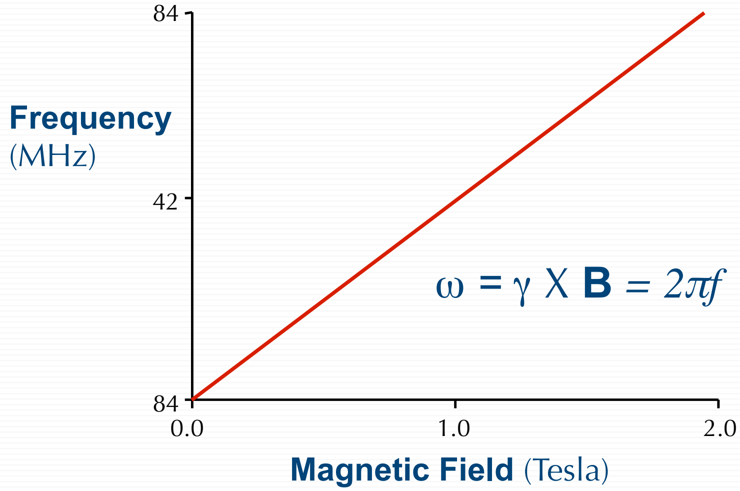
*Everything is Linear to First Order *Almost

"The faster you drive, the quicker you will get there."

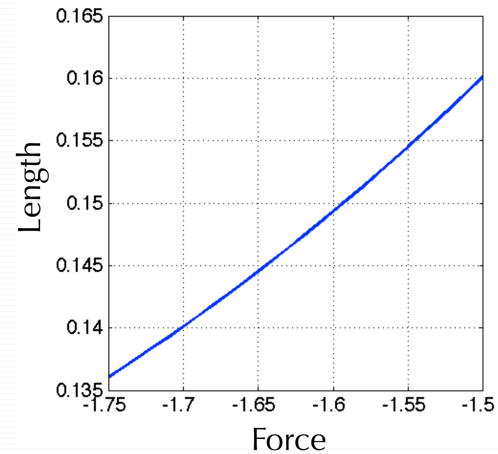
"If you pull a rubber band twice as hard, it will become twice as long."

"If you hit a ball twice as hard, it will go twice as far."

The Larmor Relation



Linearity - Examples



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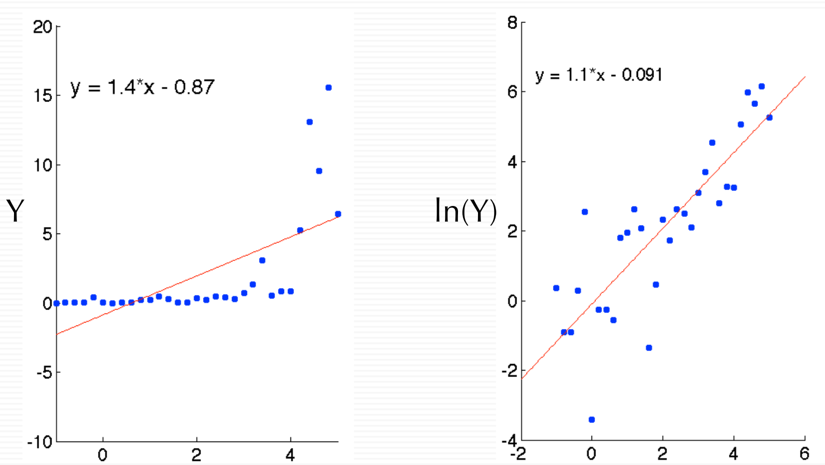
Linearization



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Moving Average

http://goldstocktrades.com/blog/wp-content/uploads/2010/09/Gld-9-20-10.jpg

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2D Convolution

Reference Figure

		1		
	1	1	1	
1	1	1	1	1
	1	1	1	
		1		

Boxcar

2D Convolution

Reference Figure

	-1	
-1	5	-1
	-1	

Laplacian

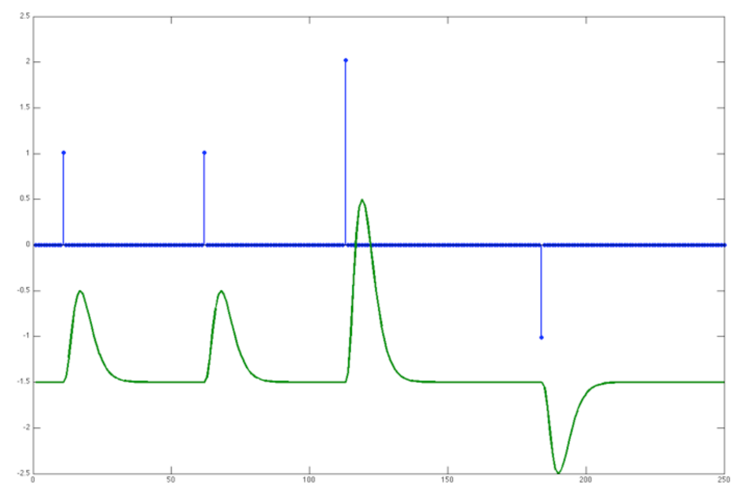
2D Convolution

Reference Figure

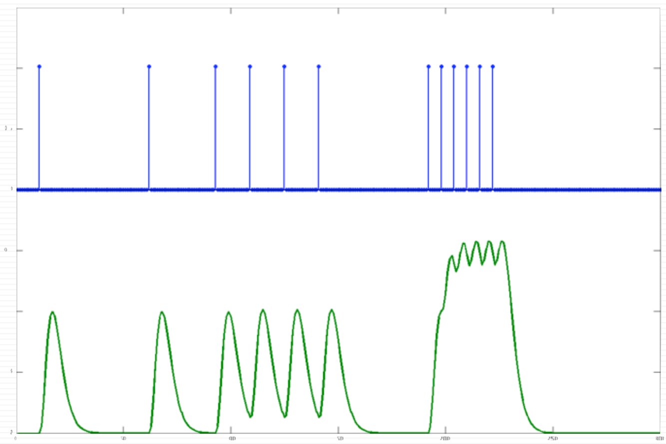
4	9	12	9	4
6	12	16	12	6
7	16	20	16	7
6	12	16	12	6
4	9	12	9	4

Gaussian

Impulse Response: *scale and shift*



Impulse Response
superposition



Impulse Responses

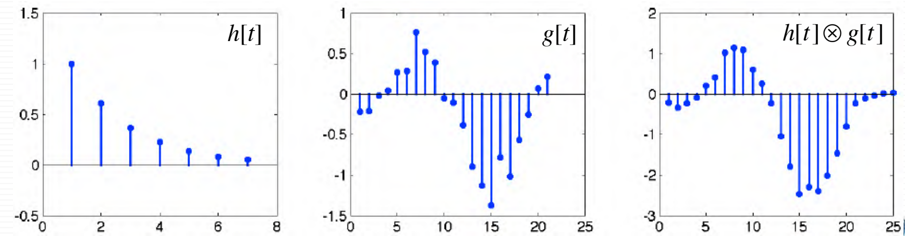
$$g[t] = g_0 + g_1[1] + g_2[2] + \ldots + g_n[n] + \ldots$$

$$h[t] = h_0 + h_1[1] + h_2[2] + \ldots + h_n[n] + \ldots$$

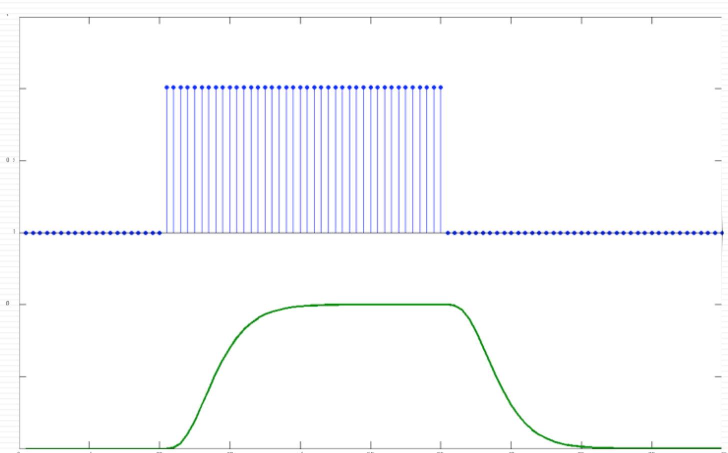
$$y[t] = g_0 h[t] + g_1[1]h[t-1] + g_2[2]h[t-2] + \ldots + g_n[n]h[t-n] + \ldots$$

$$= g[t] \otimes h[t]$$

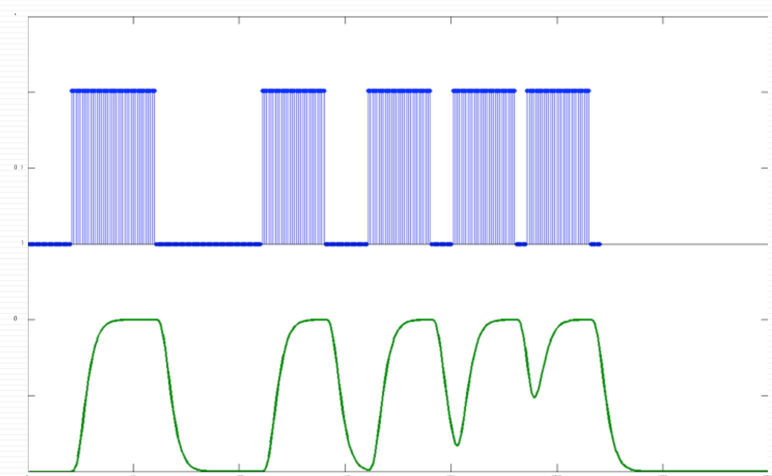
$$= \sum_{k=-\infty}^{\infty} g[k]h[n-k]$$



Rapid Events -> Blocks



Blocks and Baseline



Continuous Convolution

$$g[n]\otimes h[n]=\sum_{k=-\infty}^{\infty} g[k]h[n-k]$$

Discrete domain

$$g(t)\otimes h(t)=\int_{-\infty}^{\infty} g(\tau)h(t-\tau)d\tau$$

Continuous domain

See a great article convolution at:
http://en.wikipedia.org/wiki/Convolution

Properties of Convolution

Commutativity

$$f(t)\otimes g(t)=\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

$$u=t-\tau$$

$$=\int_{-\infty}^{\infty} f(t-u)g(u)d(t-u)$$

$$=\int_{-\infty}^{\infty} f(t-u)g(u)du$$

$$=\int_{-\infty}^{\infty} g(u)f(t-u)du$$

$$=g(t)\otimes f(t)$$

Properties of Convolution

Distributivity

$$f(t)\otimes(g(t)+h(t))=\int_{-\infty}^{\infty} f(\tau)\left[g(t-\tau)+h(t-\tau)\right]d\tau$$

$$=\int_{-\infty}^{\infty} f(\tau)g(t-\tau)+\int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$

$$=f(t)\otimes g(t)+f(t)\otimes h(t)$$

Properties of Convolution

Associativity

$$f(t)\otimes(g(t)\otimes h(t))=f(t)\otimes\int_{-\infty}^{\infty} g(\tau)h(t-\tau)d\tau$$

$$=\int_{-\infty}^{\infty} f(\varphi)\otimes\int_{-\infty}^{\infty} g(\tau)h(t-\tau-\varphi)d\tau\, d\varphi$$

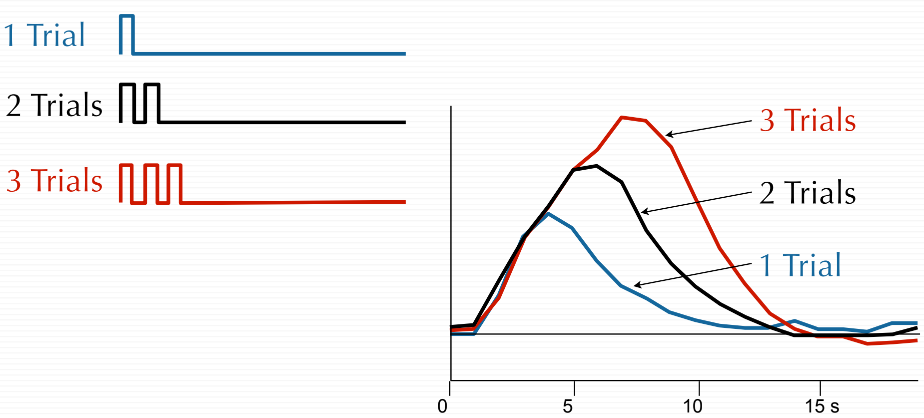
$$=\int_{-\infty}^{\infty} f(\varphi)\otimes\int_{-\infty}^{\infty} h(t-\tau-\varphi)d\varphi\big(g(\tau)d\tau\big)$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big(f(\varphi)h(t-\tau-\varphi)d\varphi\big)\otimes g(\tau)d\tau$$

$$=\int_{-\infty}^{\infty} f(\varphi)h(t-\varphi)d\varphi\otimes g(\tau)$$

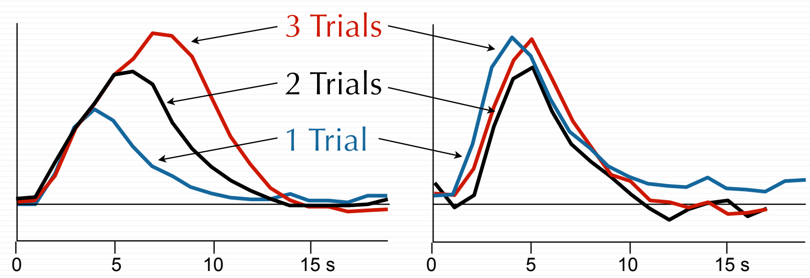
$$=\big(f(t)\otimes g(t)\big)\otimes h(t).$$

Time Invariance



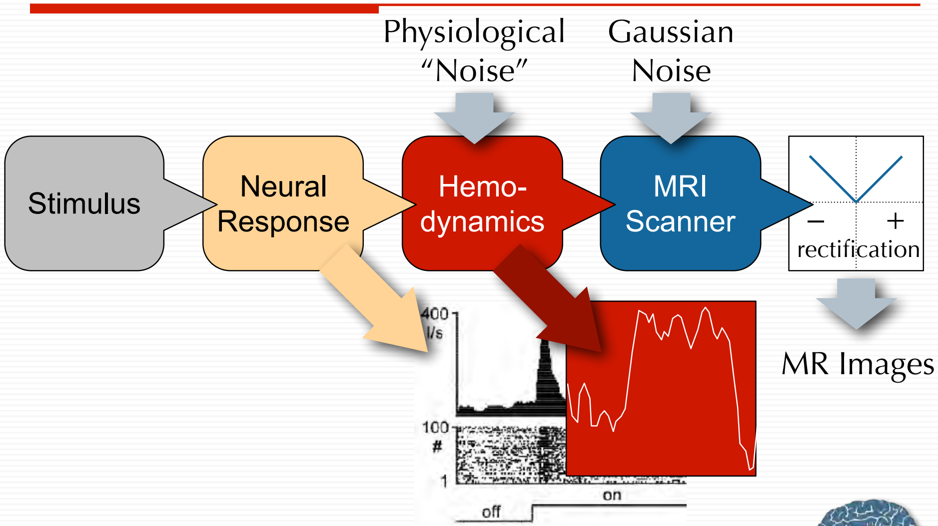
Dale and Buckner 1997

Time Invariance?

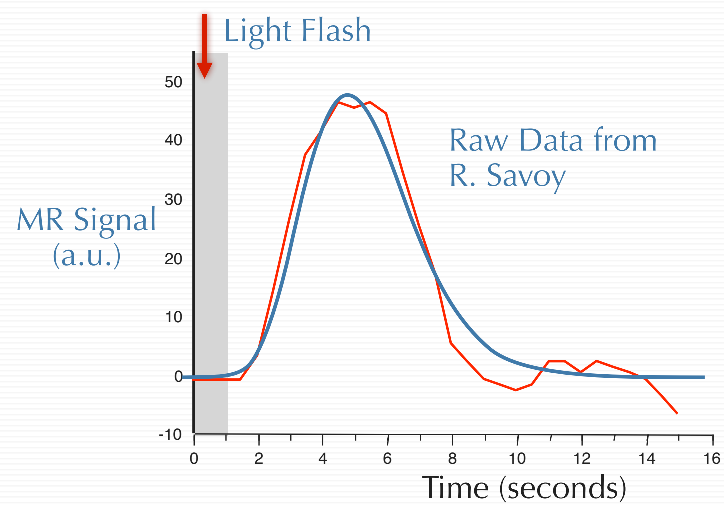


Dale and Buckner 1997

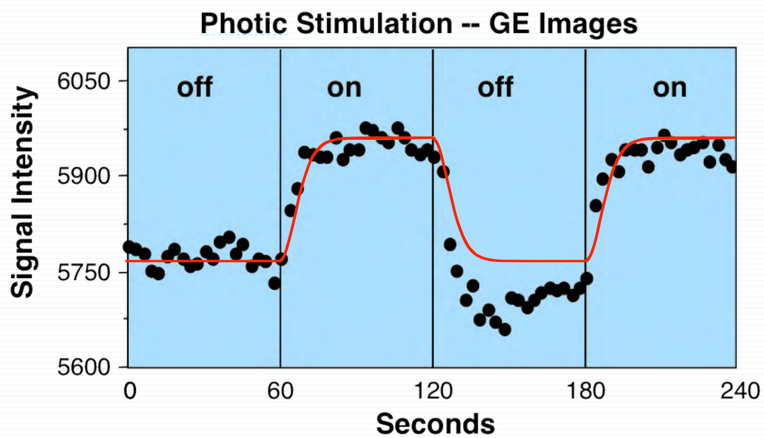
Transfer Function Model



Brain Impulse Response



Gradient-Recalled Echo



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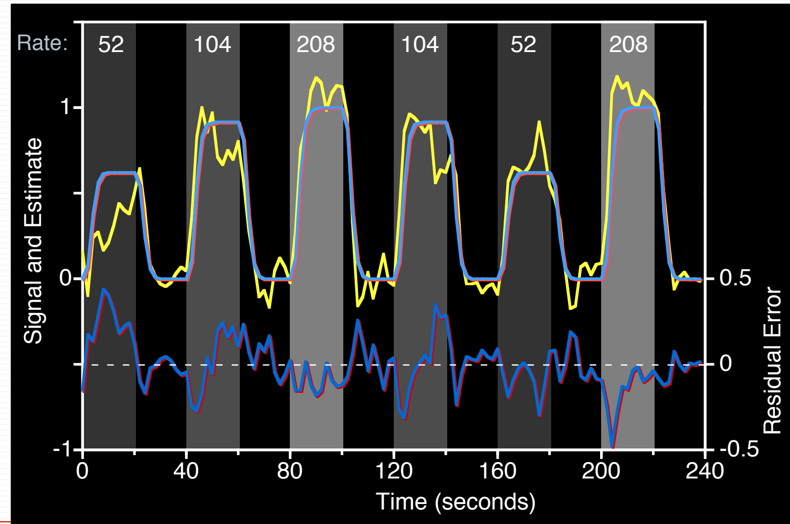
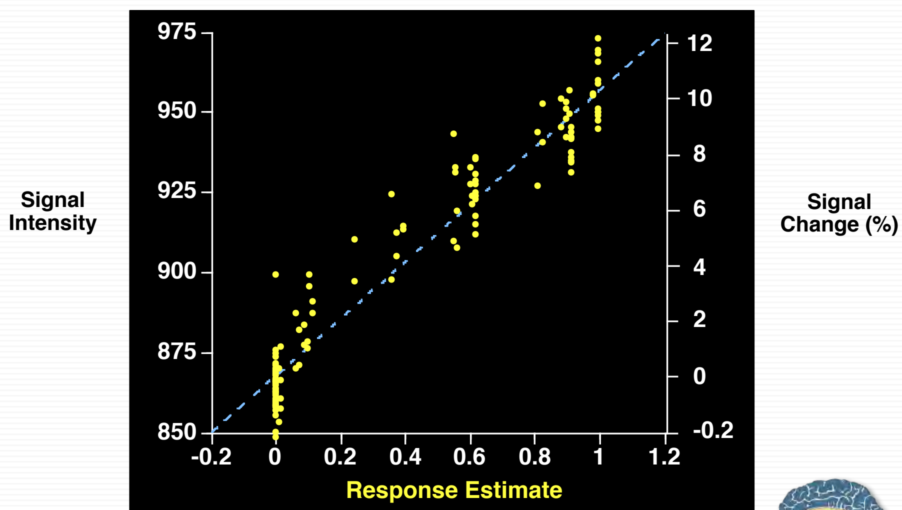
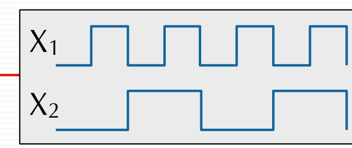
Amplitude-weighted Linear Estimate



Estimated vs. Actual *f*MRI Response



General Linear Model



(matrix form)

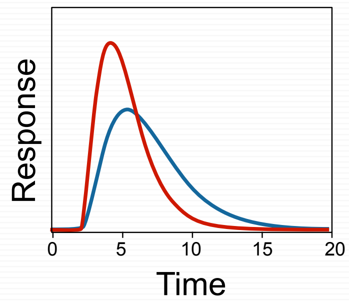
$$\mathbf{Y} = \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2 + \varepsilon$$

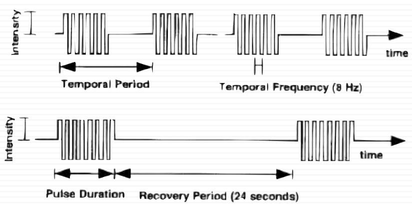
$$\mathbf{Y} = f(\mathbf{X}_1)\beta_1 + f(\mathbf{X}_2)\beta_2 + \varepsilon$$

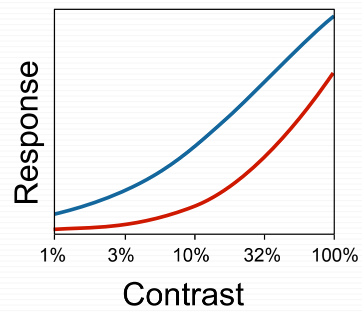
$$\mathbf{Y} = \ln(\mathbf{X}_1)\beta_1 + \ln(\mathbf{X}_2)\beta_2 + \varepsilon$$

$$\mathbf{Y} = \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2 + \mathbf{X}_1\mathbf{X}_2\beta_3 + \varepsilon$$

Time Invariance (redux)

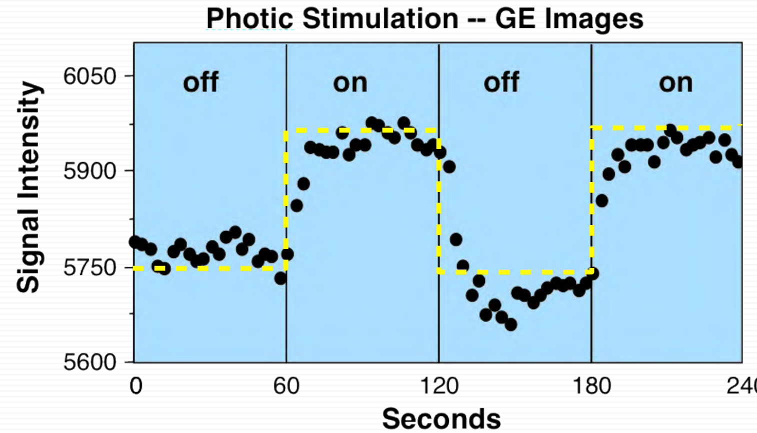






Boynton, et al., 1996

Gradient-Recalled Echo



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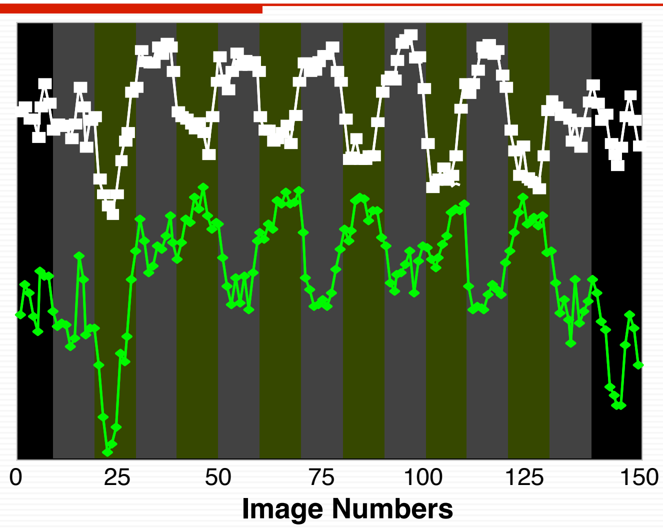
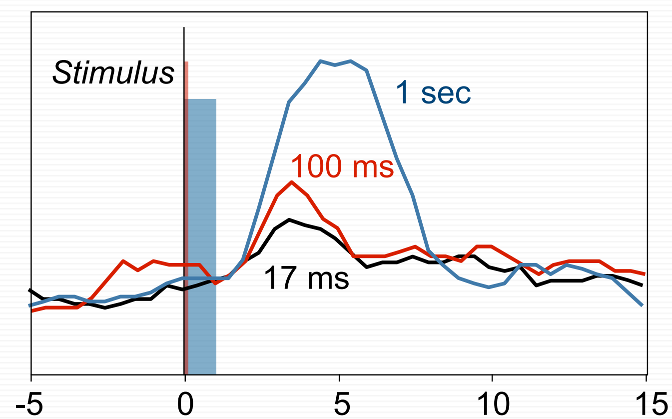
Hemifield Alternation 20 seconds

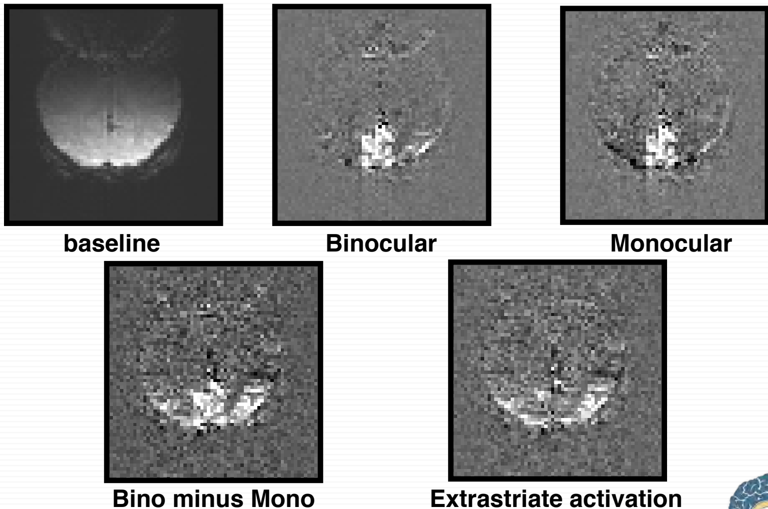


Response Latency vs. Stimulus Duration Average of 10 recordings



Data courtesy of Robert Savoy

Binocular vs Monocular Activation



The Euler Relation

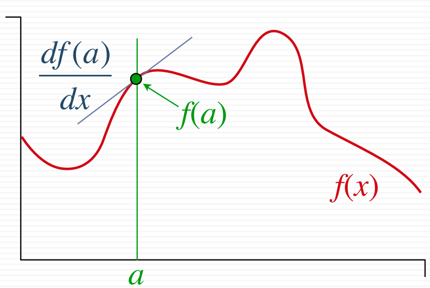
$$e^{ix} = \cos x + i \sin x$$

The Taylor Series

$$f(x) = f(a)(x-a)^0 + \frac{df(a)}{dx}\frac{(x-a)^1}{1!} + \frac{d^2 f(a)}{dx^2}\frac{(x-a)^2}{2!} + \dots + \frac{d^n f(a)}{dx^n}\frac{(x-a)^n}{n!} + \dots$$

$$= \sum_0^\infty f^{(n)} \frac{(x-a)^n}{n!}$$

$$\approx f(a) + (x-a)\frac{df}{dx}$$



The Euler Relation - getting there

Taylor Series

$$f(x) = f(a)(x-a)^0 + \frac{df(a)}{dx}\frac{(x-a)^1}{1!} + \frac{d^2 f(a)}{dx^2}\frac{(x-a)^2}{2!} + \dots + \frac{d^n f(a)}{dx^n}\frac{(x-a)^n}{n!} + \dots$$

$$= \sum_0^\infty f^{(n)}(a) \frac{(x-a)^n}{n!}$$

McLaurin Series

$$f(x) = f(0) + \frac{df(0)x}{dx} + \frac{d^2 f(0)}{dx^2}\frac{x^2}{2!} + \frac{d^3 f(0)}{dx^3}\frac{x^3}{3!} + \dots + \frac{d^n f(0)}{dx^n}\frac{x^n}{n!} + \dots$$

$$f(x) = \sum_{n=1}^\infty f^{(n)}(0)\frac{x^n}{n!}$$

The Euler Relation - getting there

$$f(x)=\sum_{n=1}^{\infty} f^{(n)}\frac{x^n}{n!}+f(0)$$

$$\sin(x)=\sin(0)+x\cos(0)-\frac{x^2\sin(0)}{2!}-\frac{x^3\cos(0)}{3!}+\frac{x^4\sin(0)}{4!}+\frac{x^5\cos(0)}{5!}+\ldots$$

$$=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$$

$$\cos(x)=\cos(0)+x\sin(0)-\frac{x^2\cos(0)}{2!}-\frac{x^3\sin(0)}{3!}+\frac{x^4\cos(0)}{4!}+\frac{x^5\sin(0)}{5!}+\ldots$$

$$=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$$

$$e^u=1+u+\frac{u^2}{2!}+\frac{u^3}{3!}+\frac{u^4}{4!}+\frac{u^5}{5!}+\frac{u^6}{6!}+\frac{u^7}{7!}+\ldots$$

The Euler Relation - getting there

$$e^u=1+u+\frac{u^2}{2!}+\frac{u^3}{3!}+\frac{u^4}{4!}+\frac{u^5}{5!}+\frac{u^6}{6!}+\frac{u^7}{7!}+\ldots$$

$$\boxed{e^{ix}}=1+ix+\frac{i^2x^2}{2!}+\frac{i^3x^3}{3!}+\frac{i^4x^4}{4!}+\frac{i^5x^5}{5!}+\frac{i^6x^6}{6!}+\frac{i^7x^7}{7!}+\ldots$$

$$=1+ix-\frac{x^2}{2!}-i\frac{x^3}{3!}+\frac{x^4}{4!}+i\frac{x^5}{5!}-\frac{x^6}{6!}-i\frac{x^7}{7!}+\ldots$$

$$=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots+i\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots\right]$$

$$\boxed{=\cos(x)+i\sin(x).}$$

Fourier transform

$$\mathscr{F}(s)=\mathscr{F}\left(f(x)\right)$$

$$=\int_{-\infty}^{\infty} f(x)\left[\cos(2\pi sx)-i\sin(2\pi sx)\right]dx$$

$$=\int_{-\infty}^{\infty} f(x)e^{-2\pi isx}\,dx$$

$$\mathscr{F}^{-1}\left(\mathscr{F}(s)\right)=f(x)$$

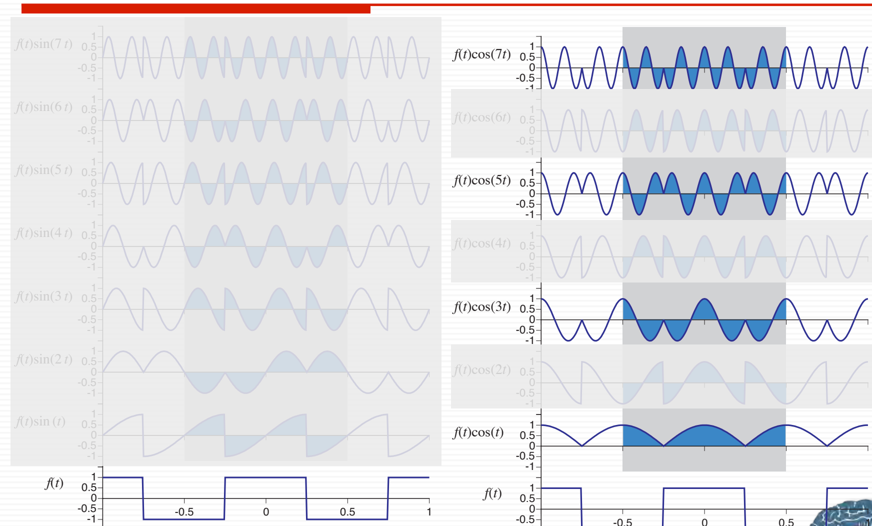
$$=\int_{-\infty}^{\infty} f(x)e^{+2\pi isx}\,dx.$$

Let $x=\dfrac{\omega}{2\pi}$:

$$\mathscr{F}(\omega)=\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$$

$$\mathscr{F}^{-1}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)e^{2\pi i\omega x}\,d\omega.$$

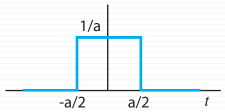
Fourier transform



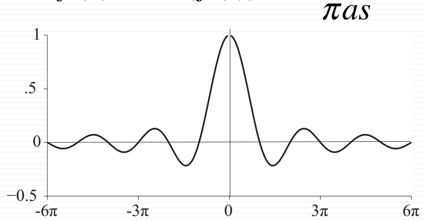
Using the Fourier Transform

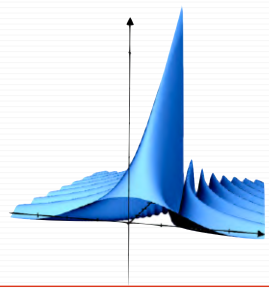
- Important Pairs - Square and sinc:

$$f(t)=\begin{cases} 1/a & \text{for } -a/2 < t < a/2 \\ 0 & \text{otherwise} \end{cases}$$



$$f(s)=\mathscr{F}(f(t))=\frac{\sin(\pi a s)}{\pi a s}$$

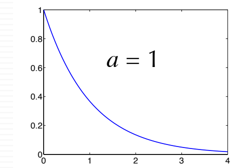




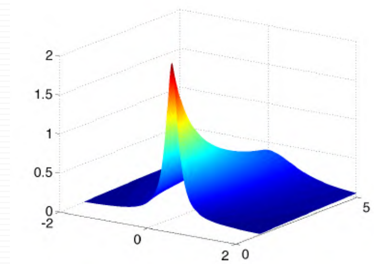
Using the Fourier Transform

- Important Pairs - Exponential and Lorenztian:

$$f(t)=e^{-a|x|}$$



$$\mathscr{F}(s)=\frac{2a}{4\pi^2 s^2+a^2}$$

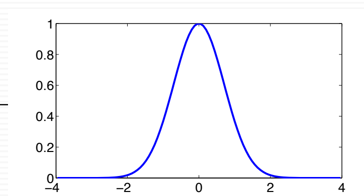


Using the Fourier Transform

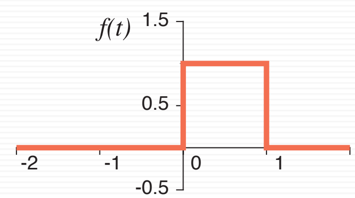
- Important Pairs - Gaussian:

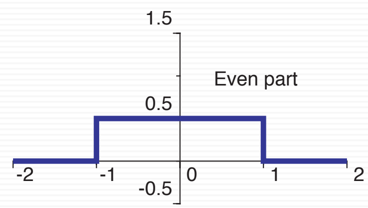
$$f(t)=e^{-ax^2}$$

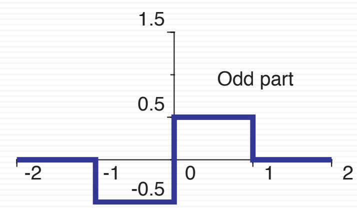
$$\mathscr{F}(s)=\frac{\sqrt{\pi}e^{-\pi x^2}}{\sqrt{a}}.$$



Odd and Even







Odd and Even

if a function is:	*its Fourier transform is:*
Real and Even	Real and Even
Real and Odd	Imaginary and Odd
Imaginary and Even	Imaginary and Even
Complex* and Even	Complex and Even
Complex and Odd	Complex and Odd
Real and Asymmetrical	Complex and Hermitian
Imaginary and Asymmetrical	Complex and anti-Hermitian
Real Even + Imaginary Odd	Real
Real Odd + Imaginary Even	Imaginary
Even	Even
Odd	Odd

Shift Theorem

$$\mathscr{F}[f(t-a)](s) = \int_{-\infty}^{\infty} f(t-a)e^{-2\pi ist}\,dt$$

$$= e^{-2\pi isa}\int_{-\infty}^{\infty} f(t-a)e^{-2\pi ist}e^{2\pi isa}\,dt$$

$$= e^{-2\pi isa}\int_{-\infty}^{\infty} f(t-a)e^{-2\pi is(t-a)}\,dt.$$

Let $u = t-a$ and $du = dt$:

$$= e^{-2\pi isa}\int_{-\infty}^{\infty} f(u)e^{-2\pi isu}du$$

$$= e^{-2\pi isa}\mathbf{F}[f(t)](s)$$

$$= (\cos(2\pi sa) - i\sin(2\pi sa)F[f(t)](s)$$

Convolution Theorem

$$\mathscr{F}(f(x)\otimes g(x)) = \mathscr{F}(f(x))\mathscr{F}(g(x))$$

$$\mathscr{F}(f(x)g(x)) = \mathscr{F}(f(x))\otimes\mathscr{F}(g(x)).$$