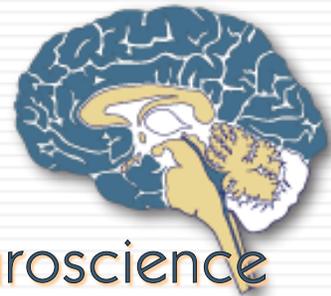


Linearity / Time Invariance

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Biomedical Physics. Biomedical Engineering



Linearity

- A system is *linear* if and only if:

$$f(Ax) = Af(x)$$

- It is time time invariant if

$$y(t) = f(x(t)) \text{ and}$$

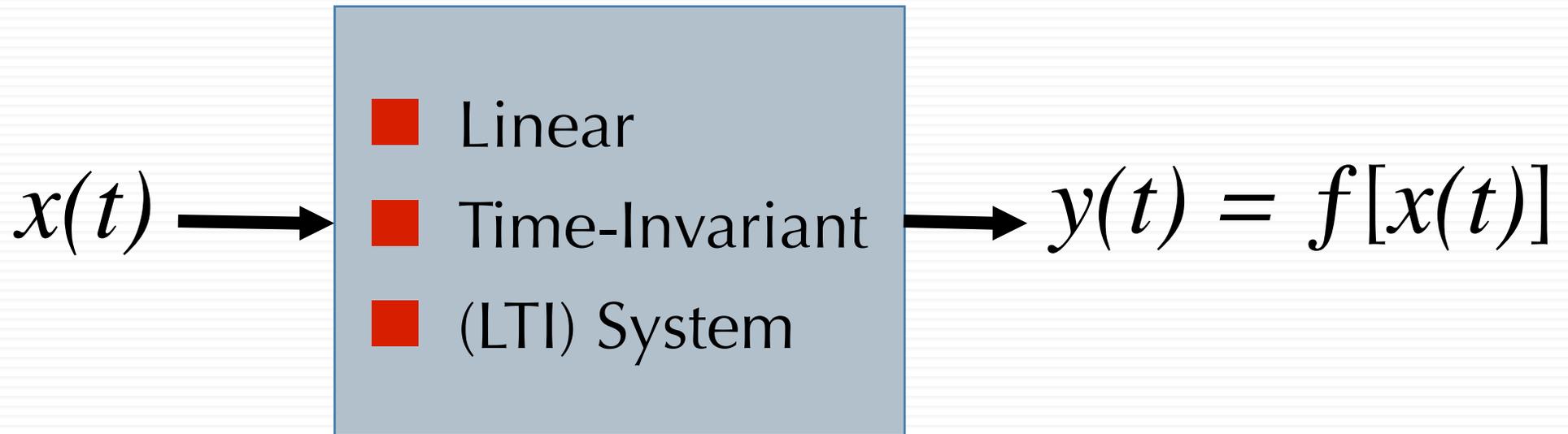
$$y(t + \delta) = f(x(t + \delta))$$

for all δ .

Linear Systems Approach

In an LTI system, given two inputs A & B:

$$f(A + B) = f(A) + f(B)$$



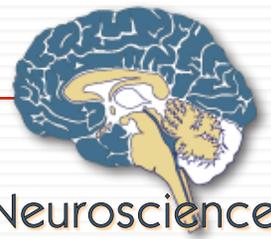
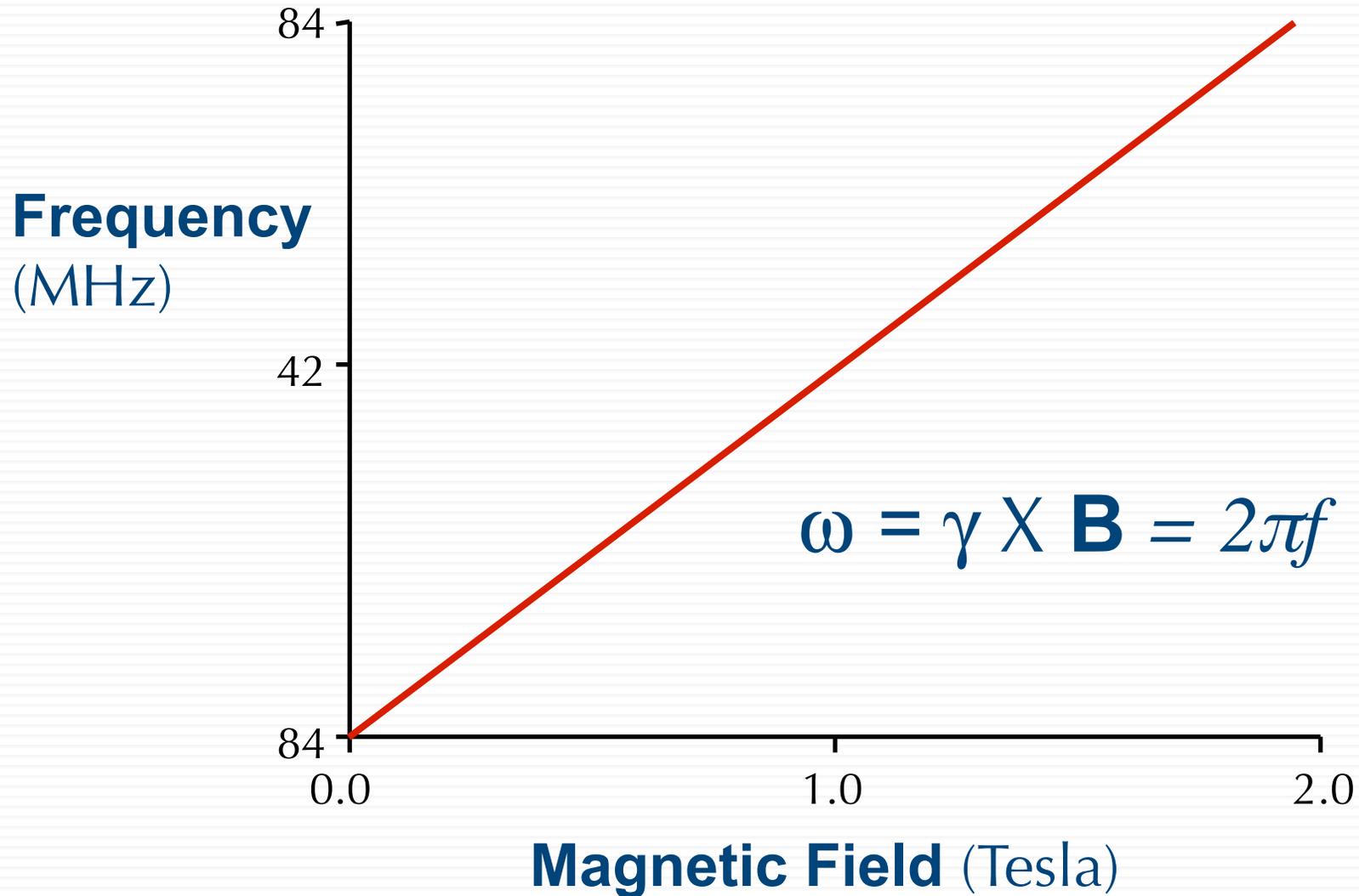
*Everything is Linear to First Order *Almost

“The faster you drive, the quicker you will get there.”

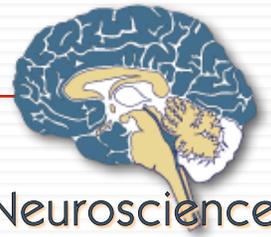
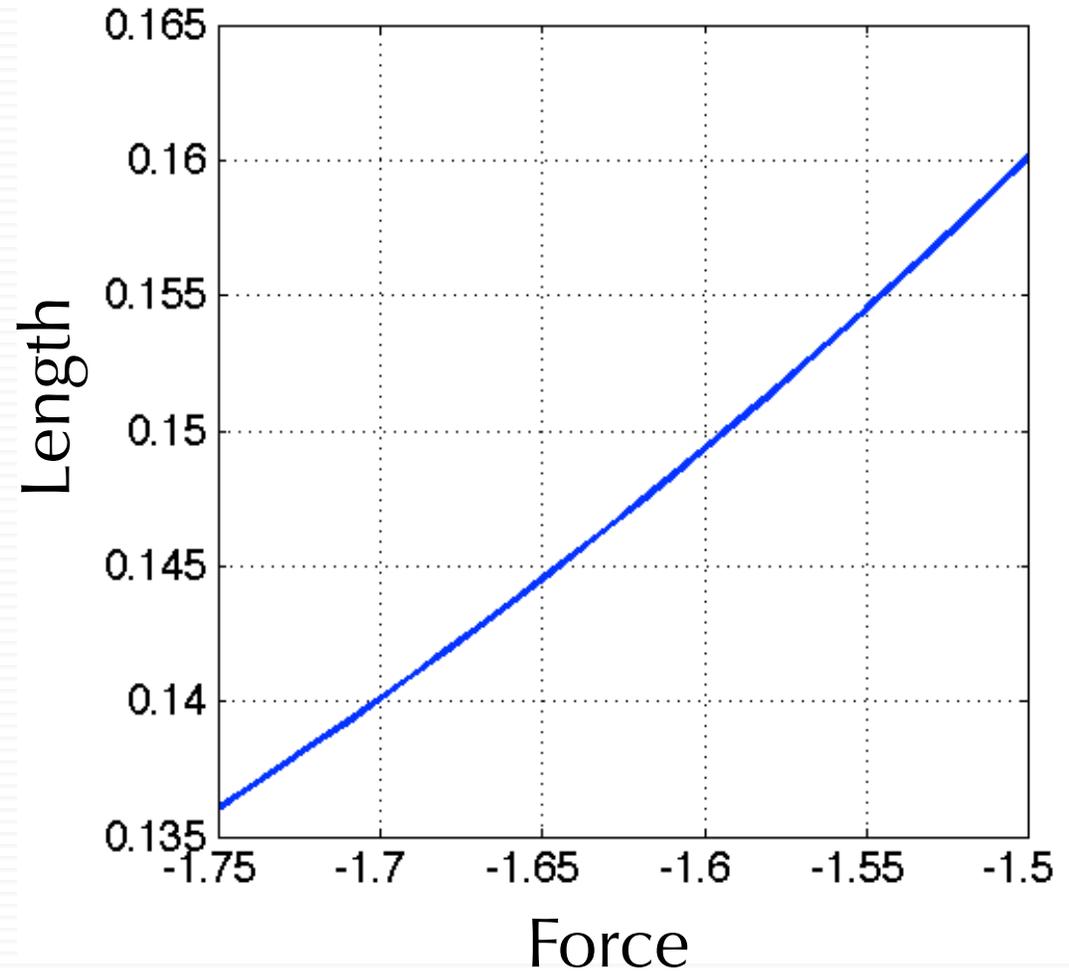
“If you pull a rubber band twice as hard, it will become twice as long.”

“If you hit a ball twice as hard, it will go twice as far.”

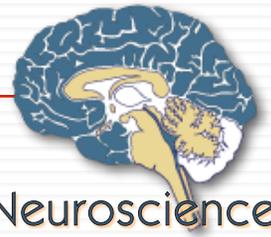
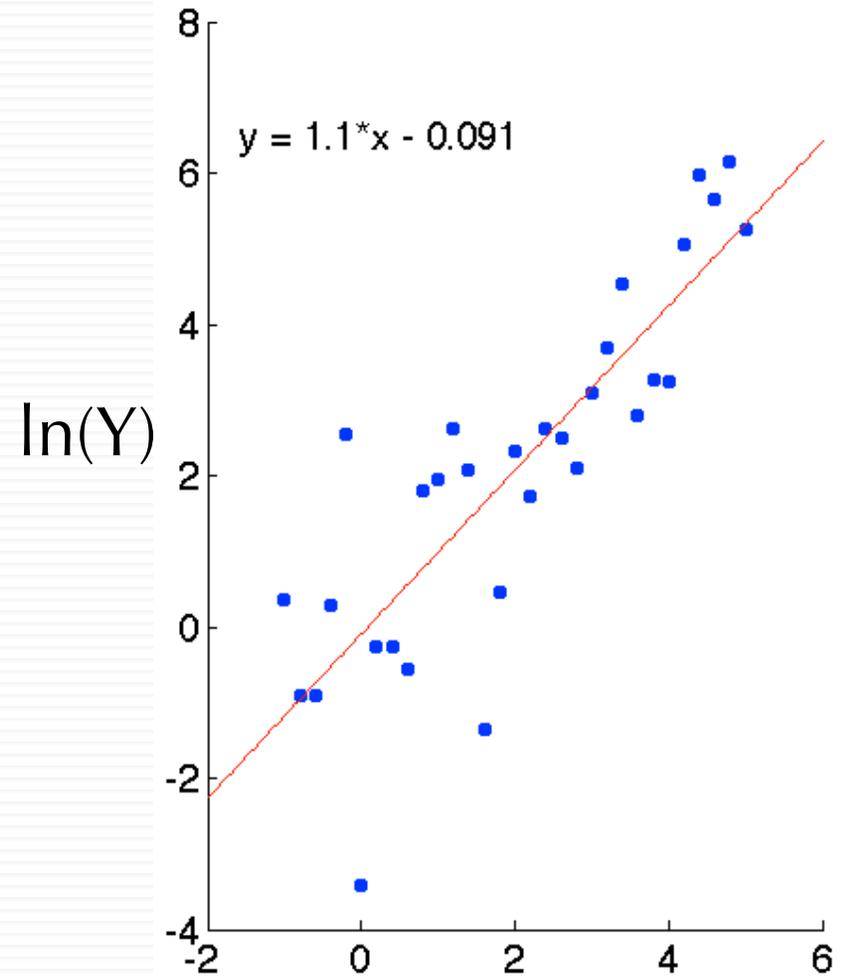
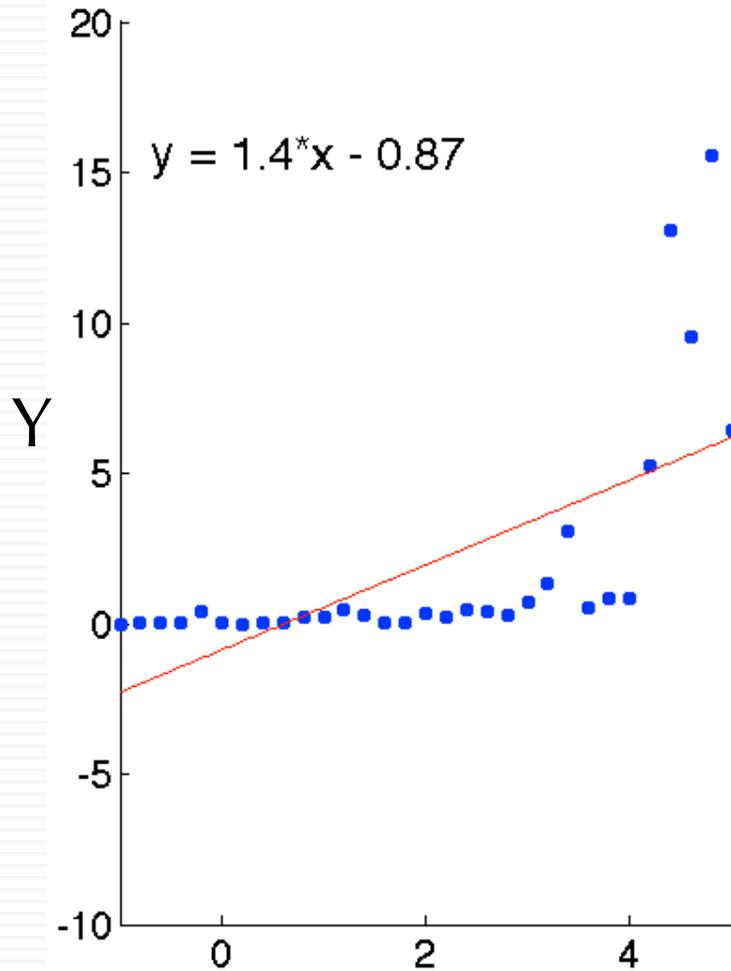
The Larmor Relation



Linearity - Examples



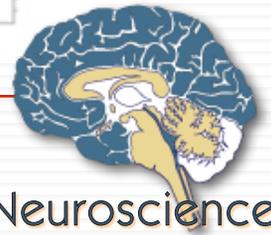
Linearization



Moving Average



<http://goldstocktrades.com/blog/wp-content/uploads/2010/09/Gld-9-20-10.jpg>



2D Convolution



Reference Figure



		1		
	1	1	1	
1	1	1	1	1
	1	1	1	
		1		

Boxcar

2D Convolution



Reference Figure



	-1	
-1	5	-1
	-1	

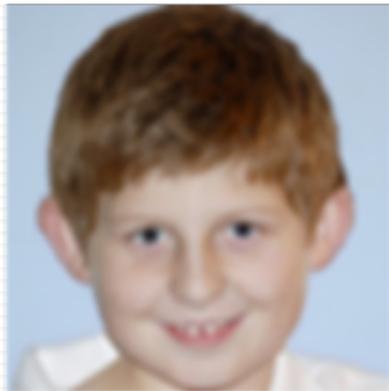
Laplacian



2D Convolution



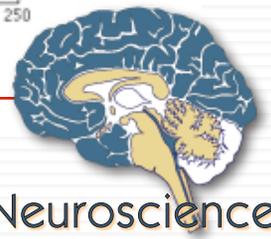
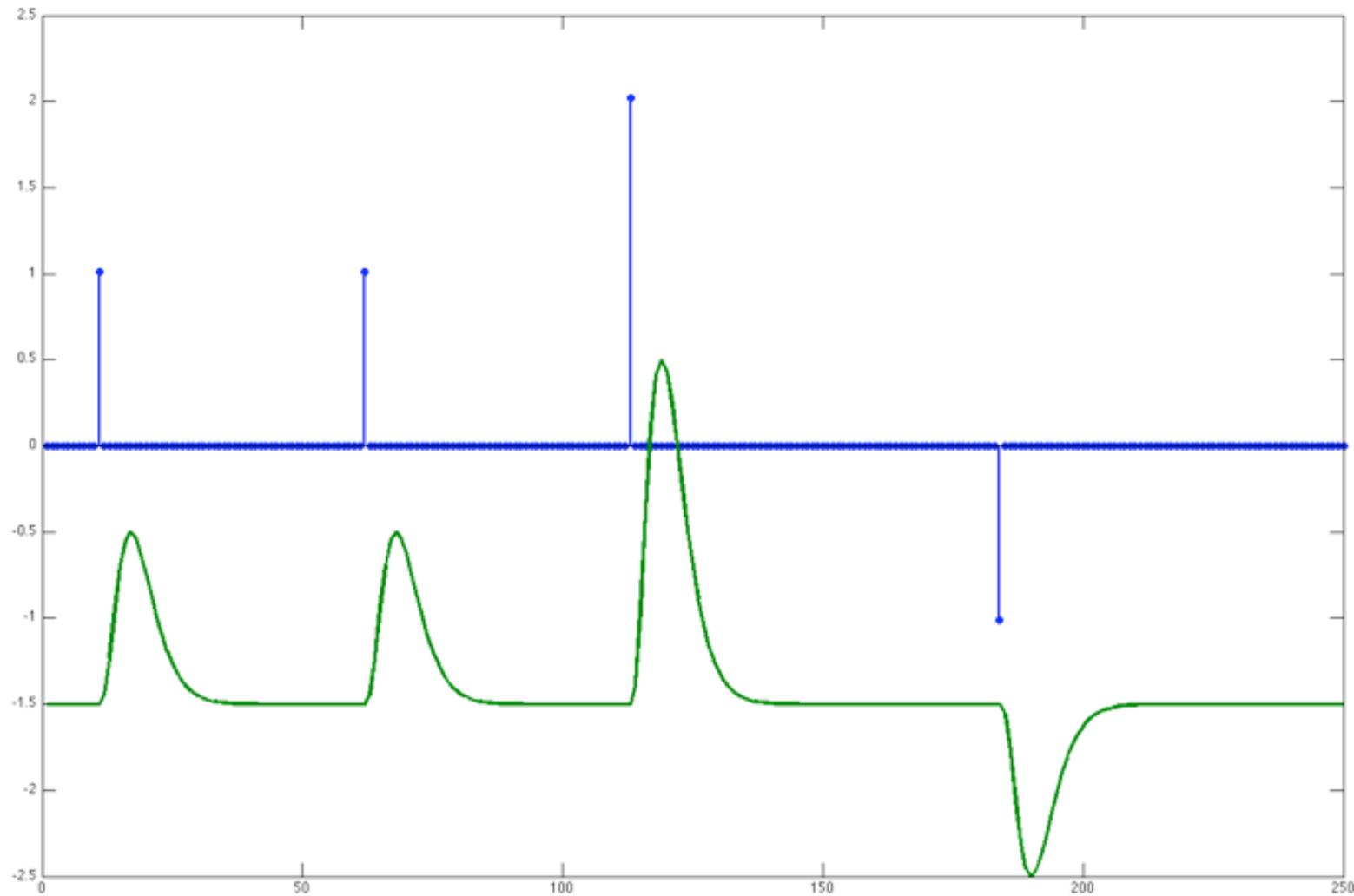
Reference Figure



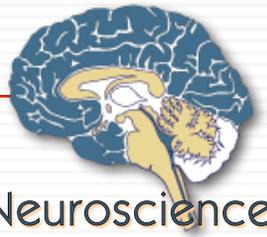
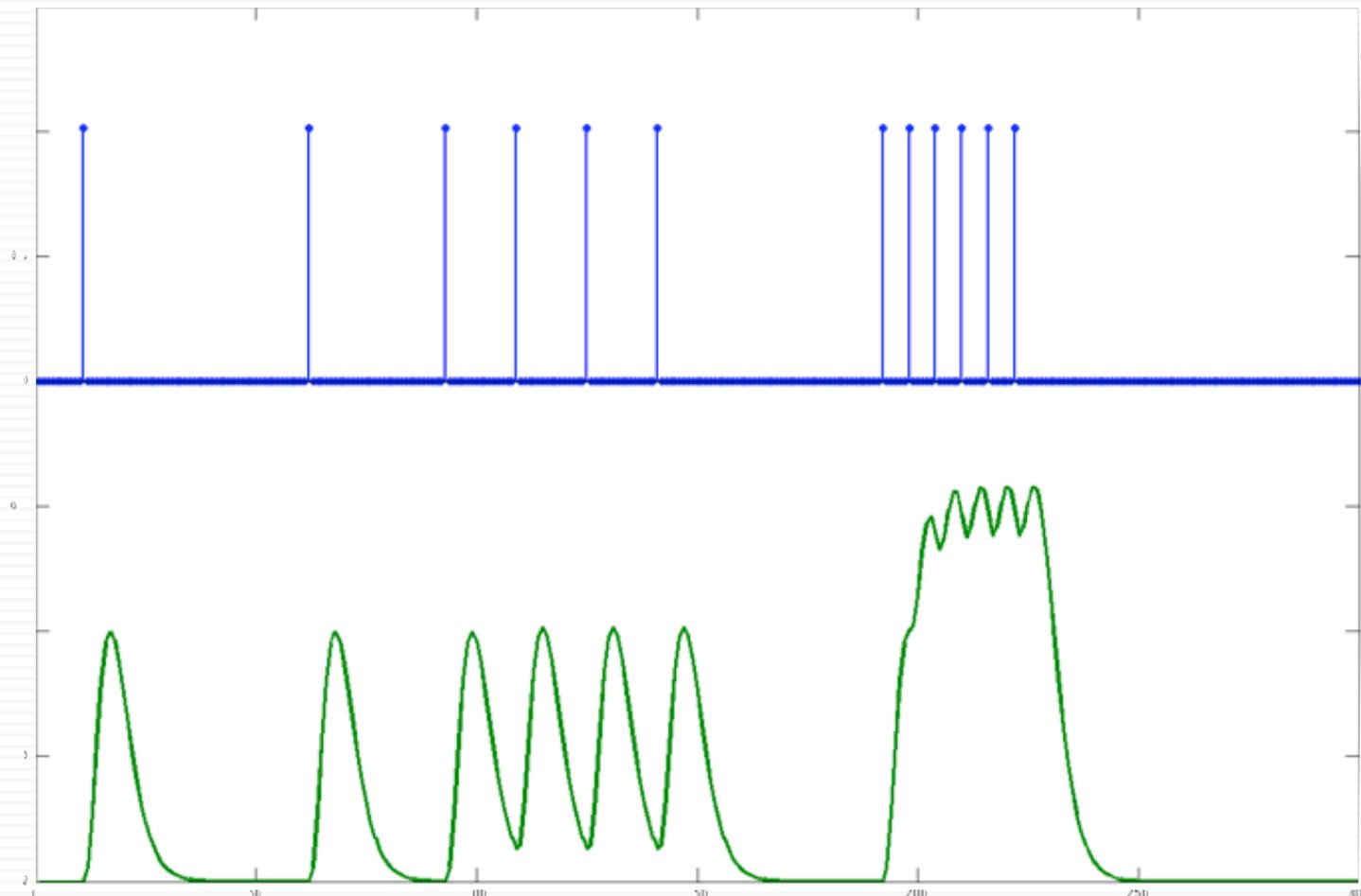
4	9	12	9	4
6	12	16	12	6
7	16	20	16	7
6	12	16	12	6
4	9	12	9	4

Gaussian

Impulse Response: *scale and shift*

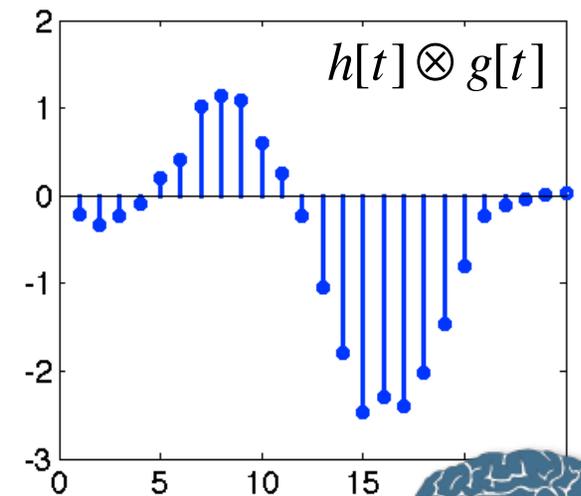
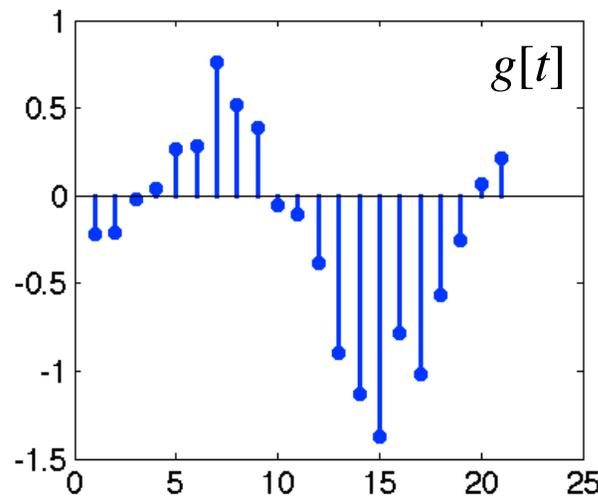
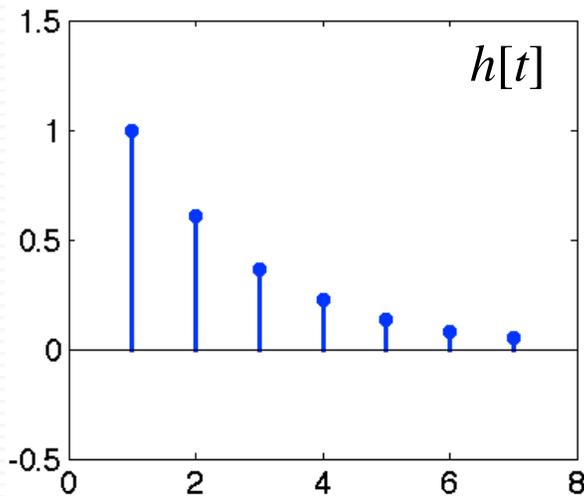


Impulse Response *superposition*

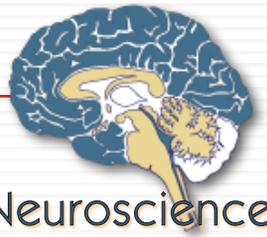
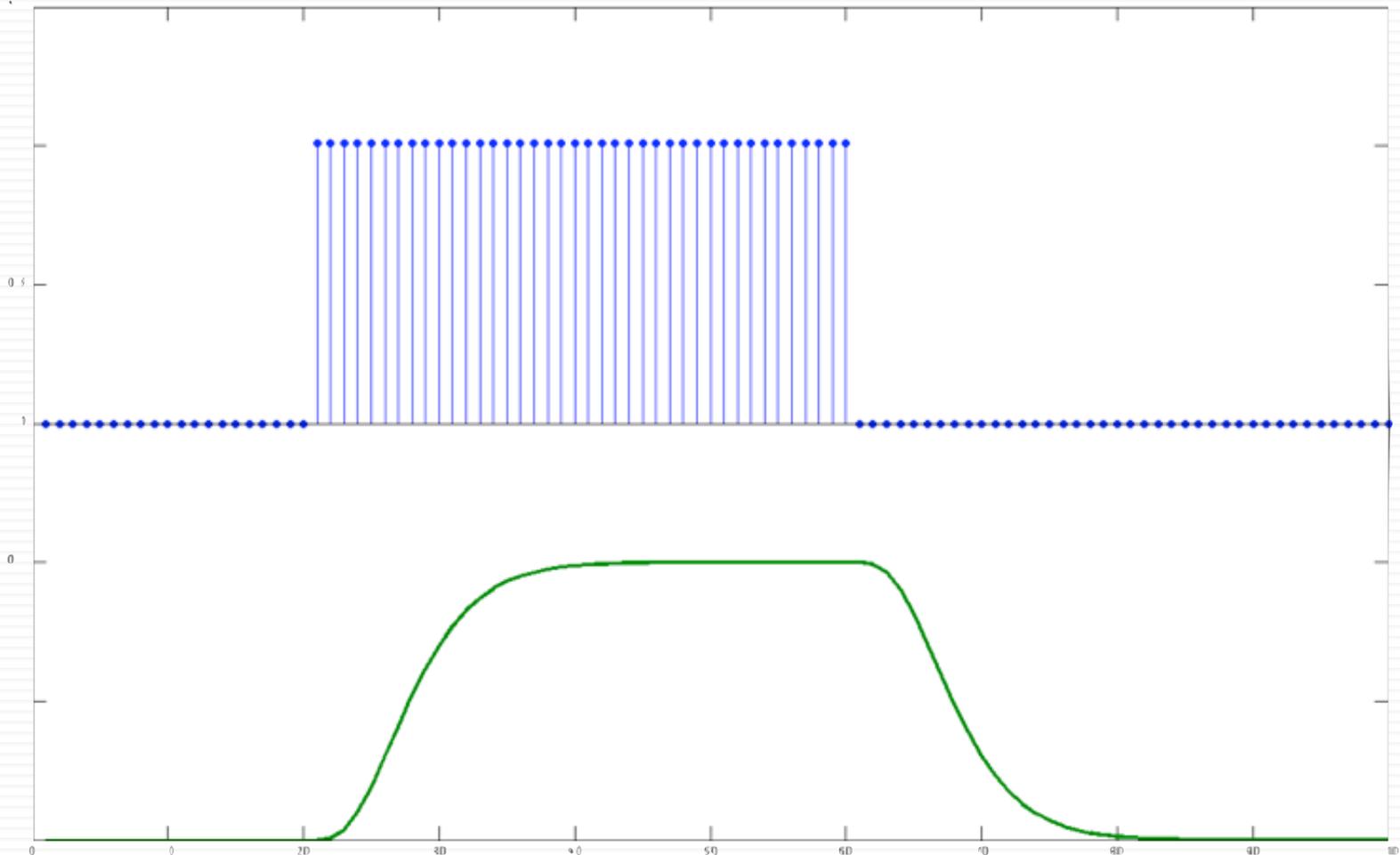


Impulse Responses

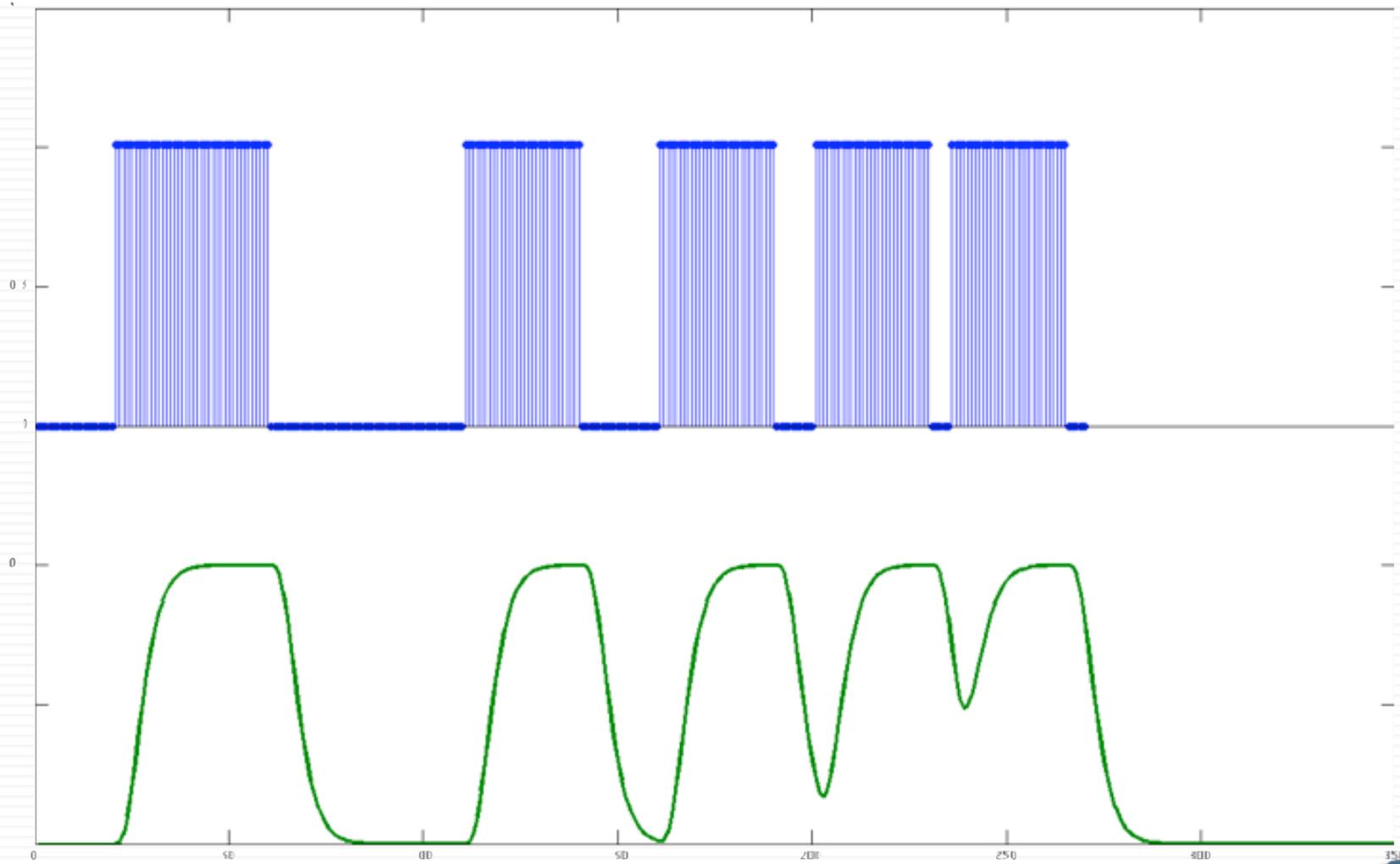
$$g[n] \otimes h[n] = \sum_{k=-\infty}^{\infty} g[k]h[n-k]$$



Rapid Events -> Blocks



Blocks and Baseline



Continuous Convolution

$$g[n] \otimes h[n] = \sum_{k=-\infty}^{\infty} g[k]h[n - k]$$

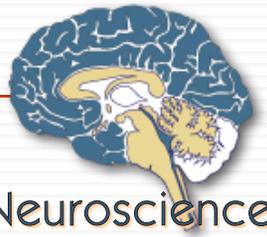
Discrete domain

$$g(t) \otimes h(t) = \int_{-\infty}^{\infty} g(\tau)h(t - \tau)d\tau$$

Continuous domain

See a great article convolution at:

<http://en.wikipedia.org/wiki/Convolution>



Properties of Convolution

Commutativity

$$\begin{aligned}f(t) \otimes g(t) &= \int_{-\infty}^{\infty} f(\tau)g(t - \tau)d\tau \\ &\quad u = t - \tau \\ &= \int_{-\infty}^{\infty} f(t - u)g(u)d(t - u) \\ &= \int_{-\infty}^{\infty} f(t - u)g(u)du \\ &= \int_{-\infty}^{\infty} g(u)f(t - u)du \\ &= g(t) \otimes f(t)\end{aligned}$$

Properties of Convolution

Distributivity

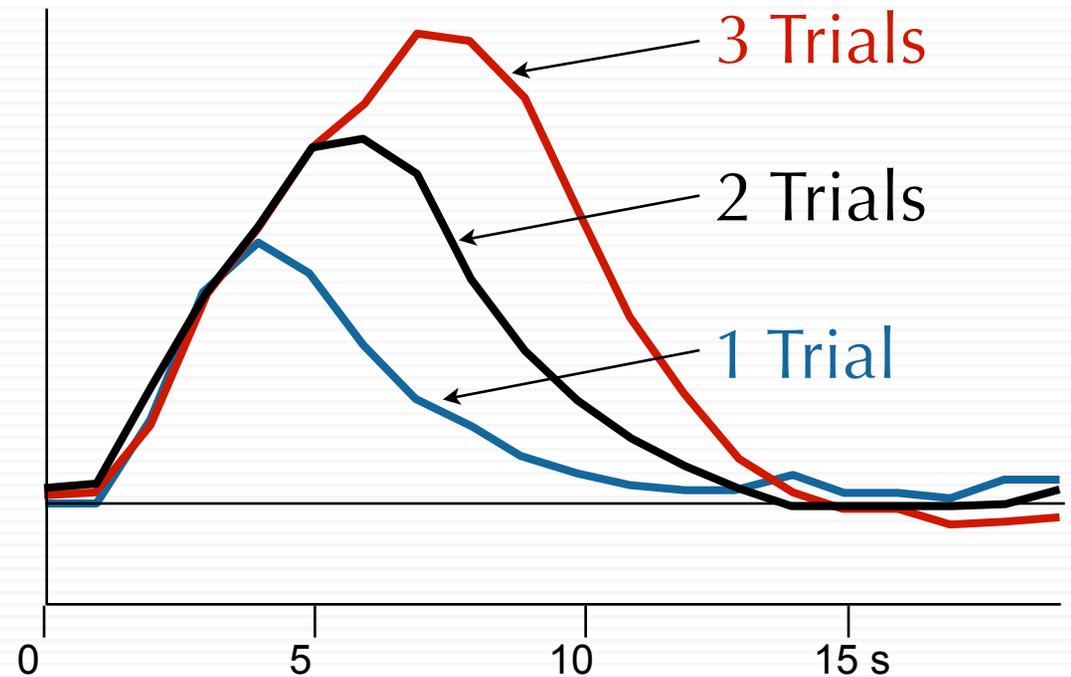
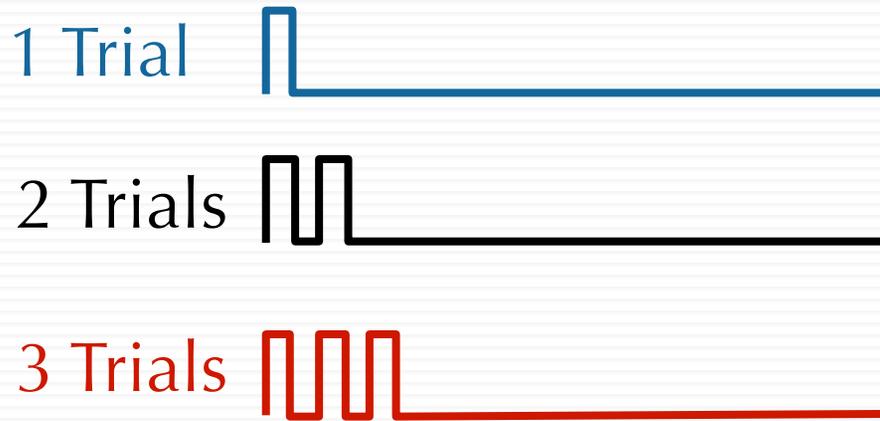
$$\begin{aligned}f(t) \otimes (g(t) + h(t)) &= \int_{-\infty}^{\infty} f(\tau) [g(t - \tau) + h(t - \tau)] d\tau \\&= \int_{-\infty}^{\infty} f(\tau) g(t - \tau) + \int_{-\infty}^{\infty} f(\tau) h(t - \tau) d\tau \\&= f(t) \otimes g(t) + f(t) \otimes h(t)\end{aligned}$$

Properties of Convolution

Associativity

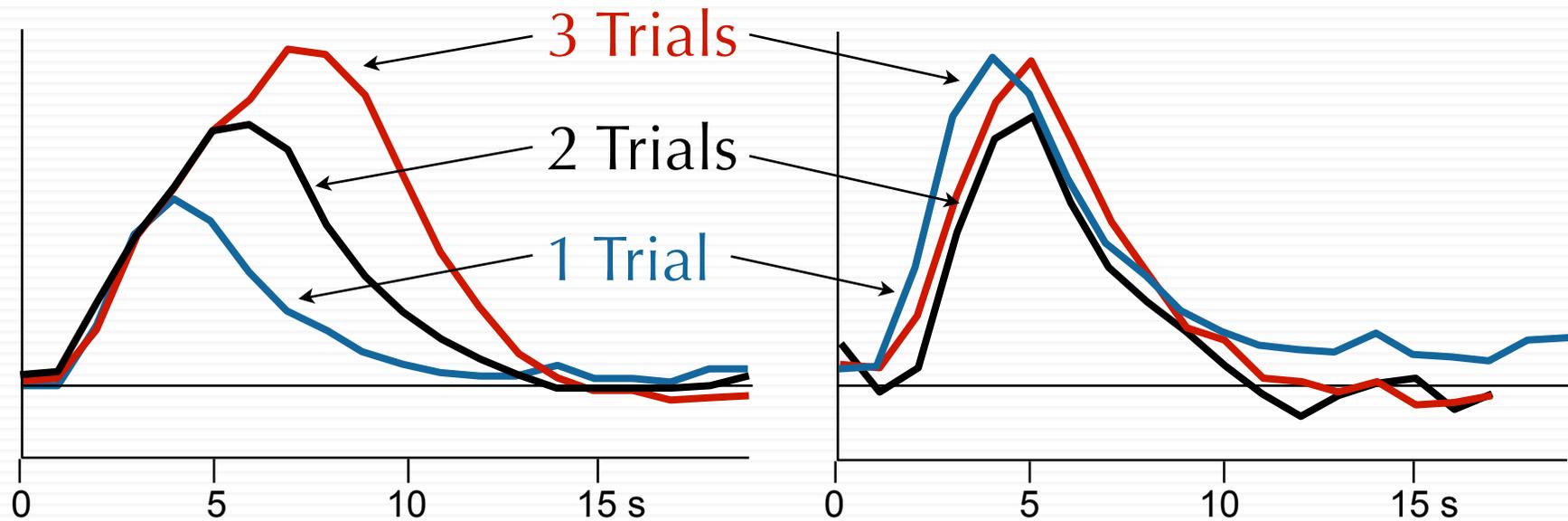
$$\begin{aligned} f(t) \otimes (g(t) \otimes h(t)) &= f(t) \otimes \int_{-\infty}^{\infty} g(\tau)h(t - \tau) d\tau \\ &= \int_{-\infty}^{\infty} f(\varphi) \otimes \int_{-\infty}^{\infty} g(\tau)h(t - \tau - \varphi) d\tau d\varphi \\ &= \int_{-\infty}^{\infty} f(\varphi) \otimes \int_{-\infty}^{\infty} h(t - \tau - \varphi) d\varphi (g(\tau) d\tau) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f(\varphi)h(t - \tau - \varphi) d\varphi) \otimes g(\tau) d\tau \\ &= \int_{-\infty}^{\infty} f(\varphi)h(t - \varphi) d\varphi \otimes g(\tau) \\ &= (f(t) \otimes g(t)) \otimes h(t). \end{aligned}$$

Time Invariance

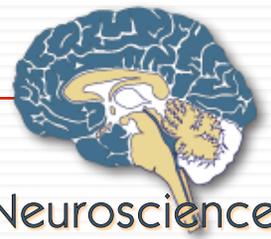


Dale and Buckner 1997

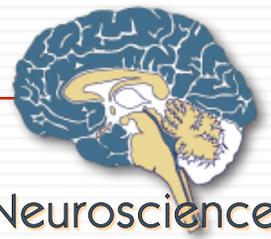
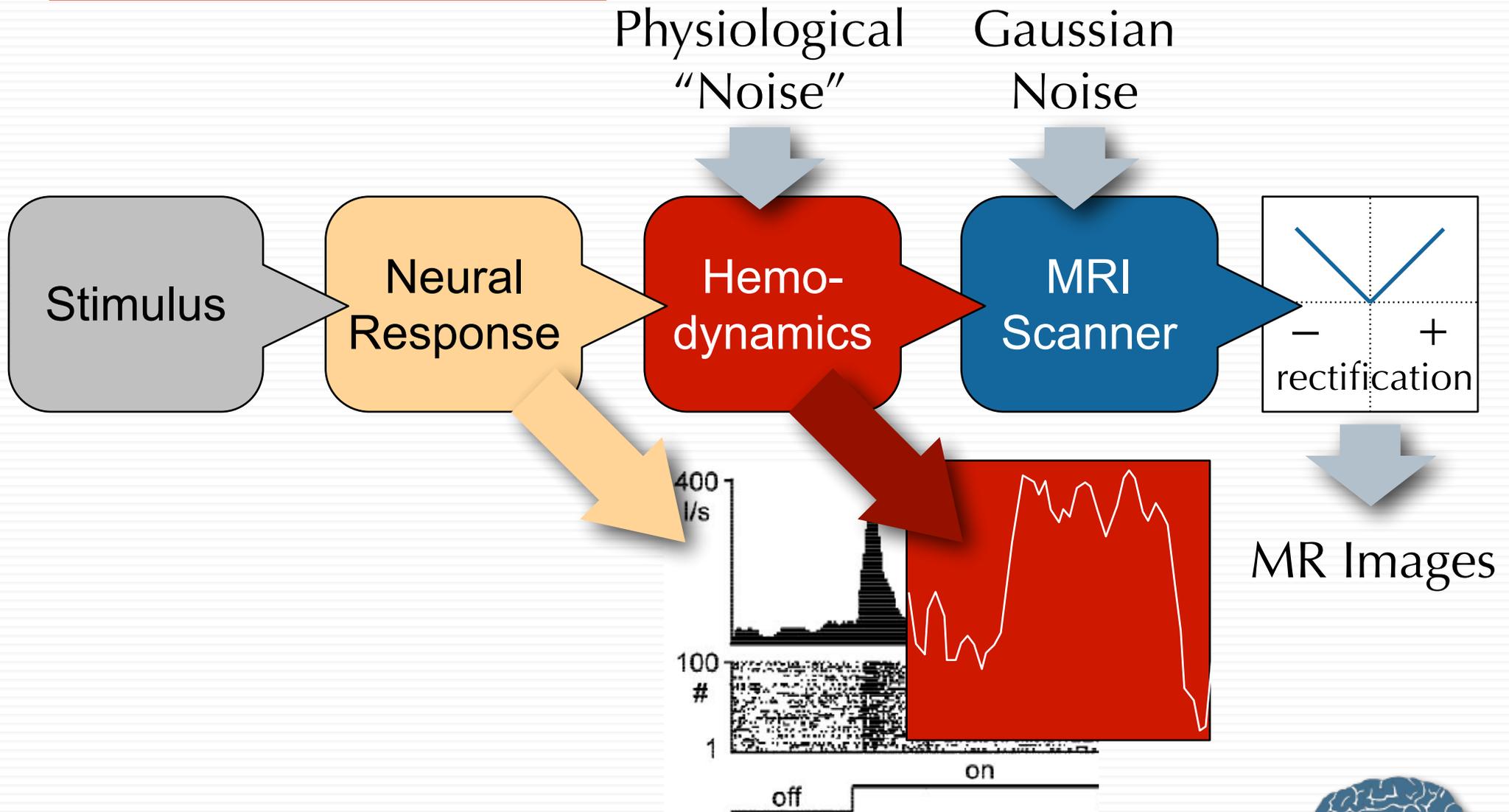
Time Invariance?



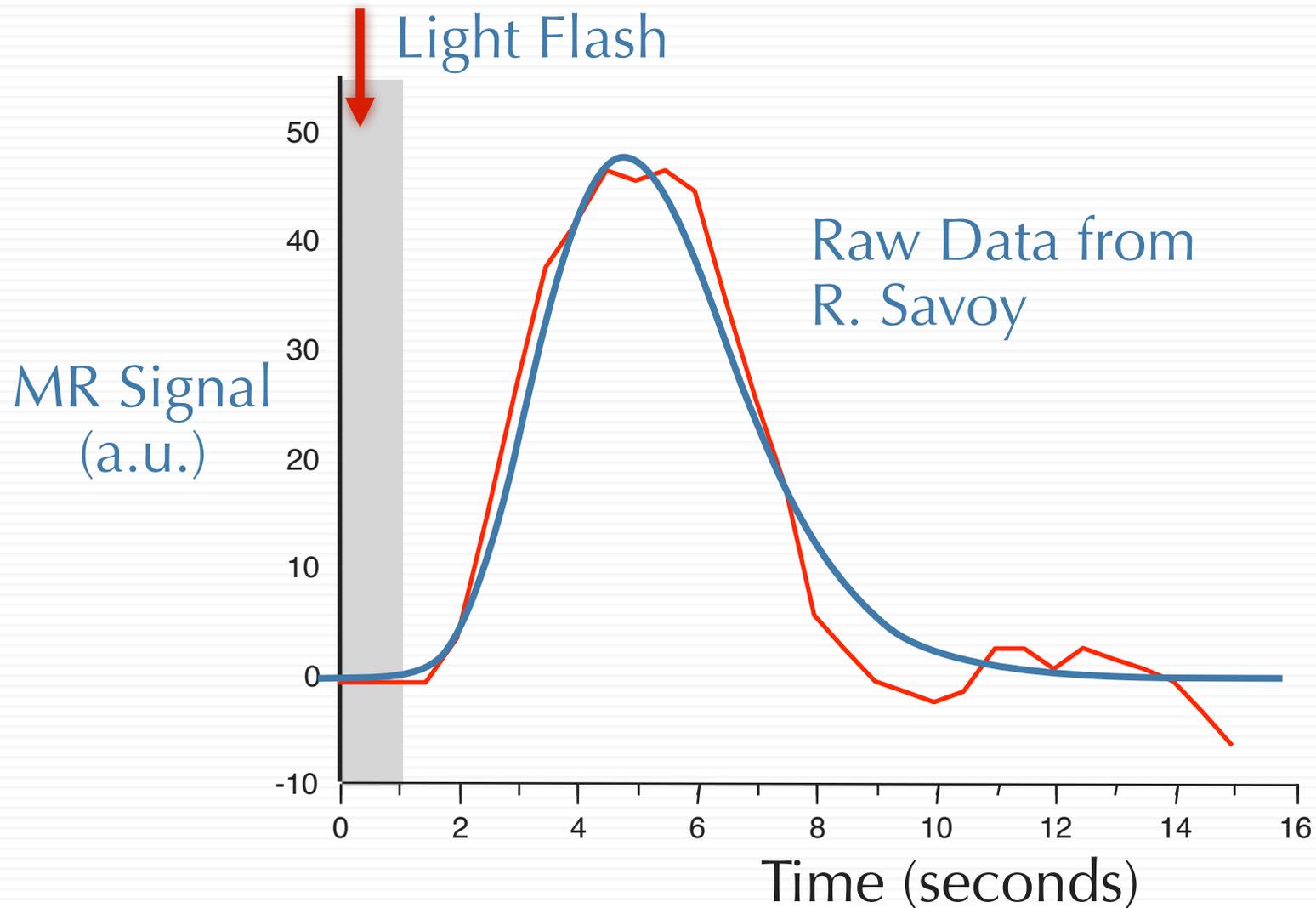
Dale and Buckner 1997



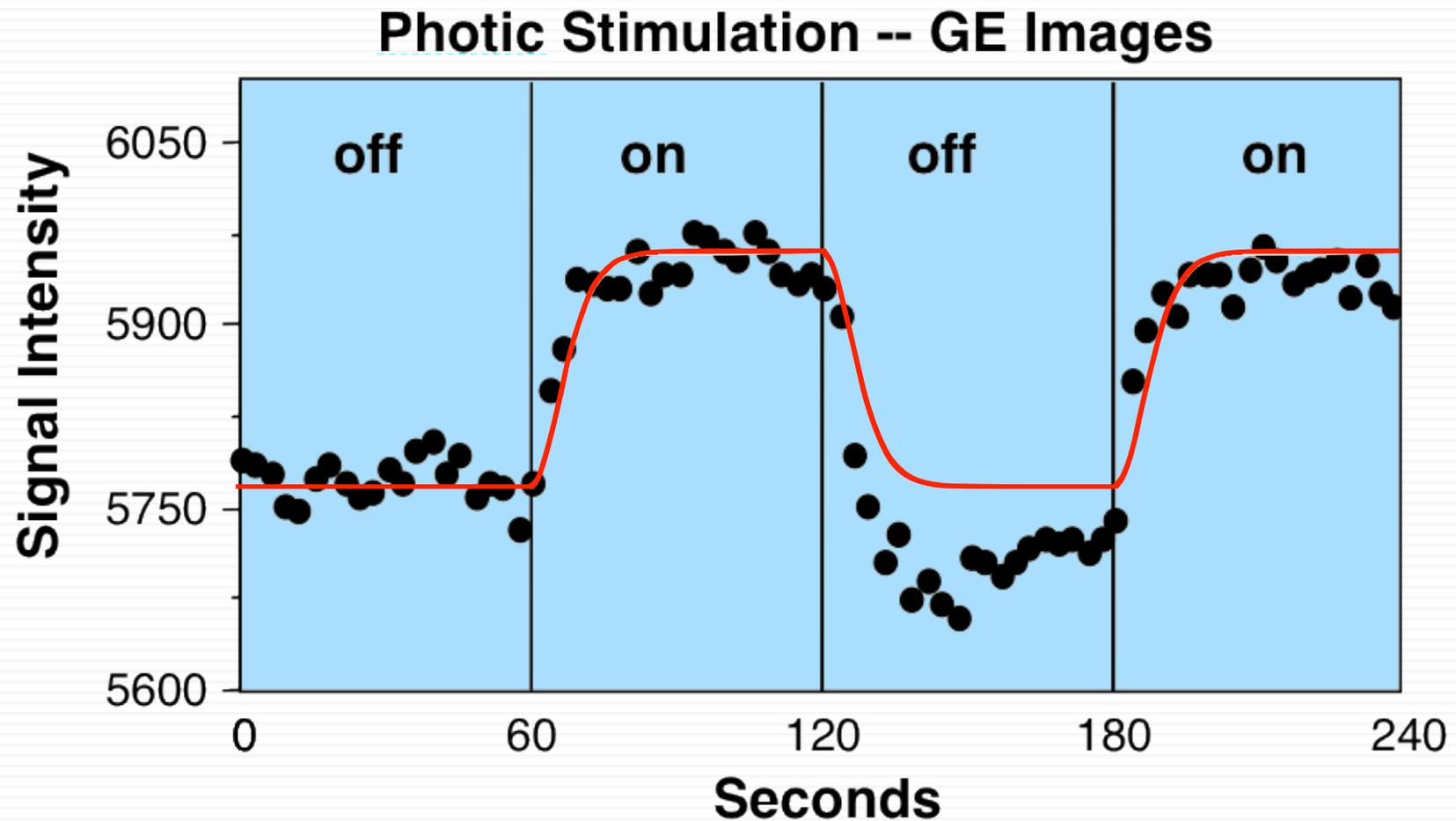
Transfer Function Model



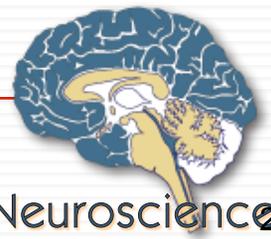
Brain Impulse Response



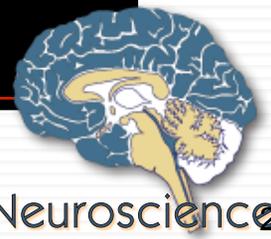
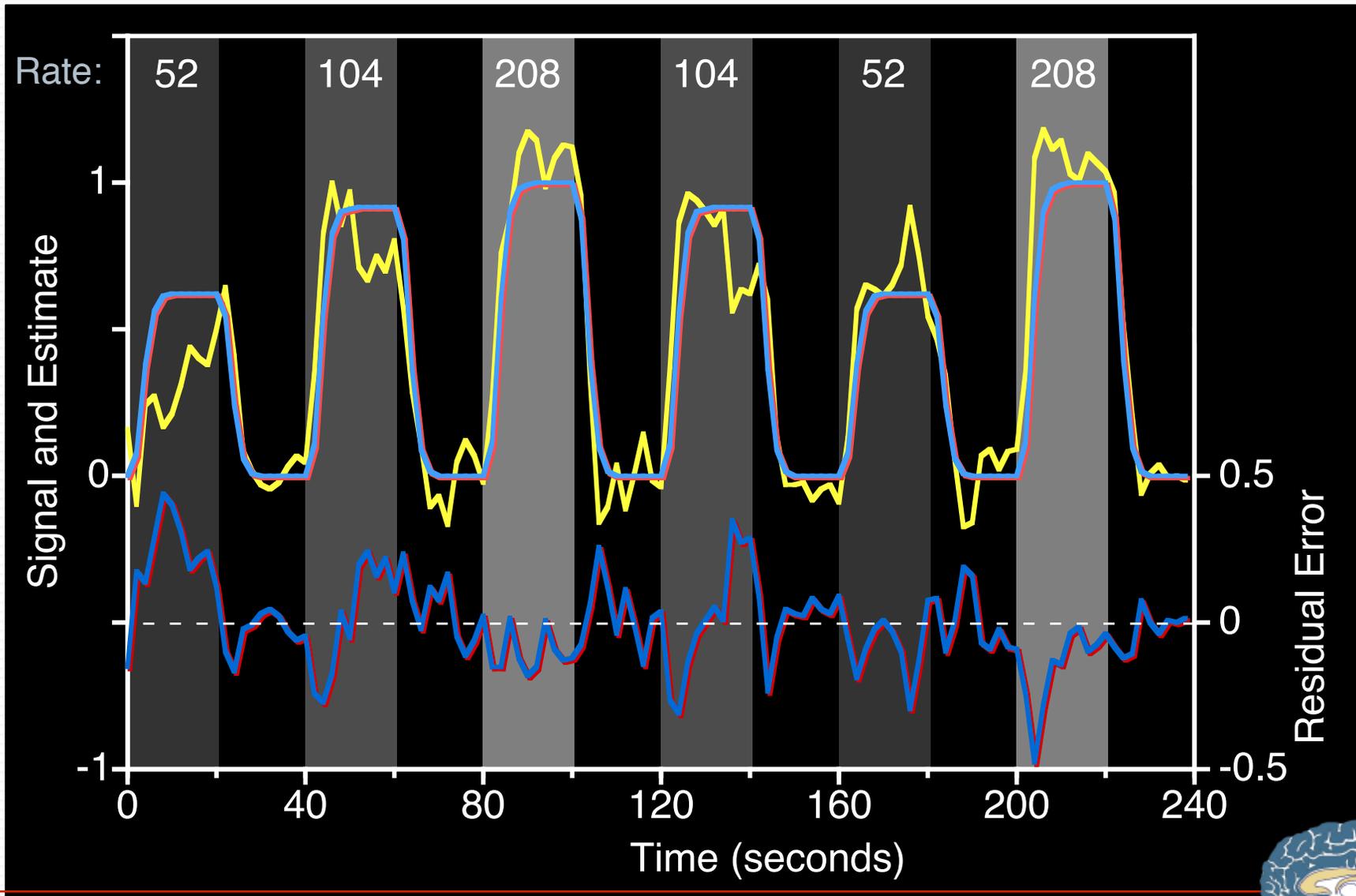
Gradient-Recalled Echo



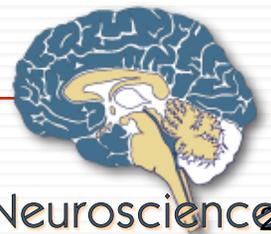
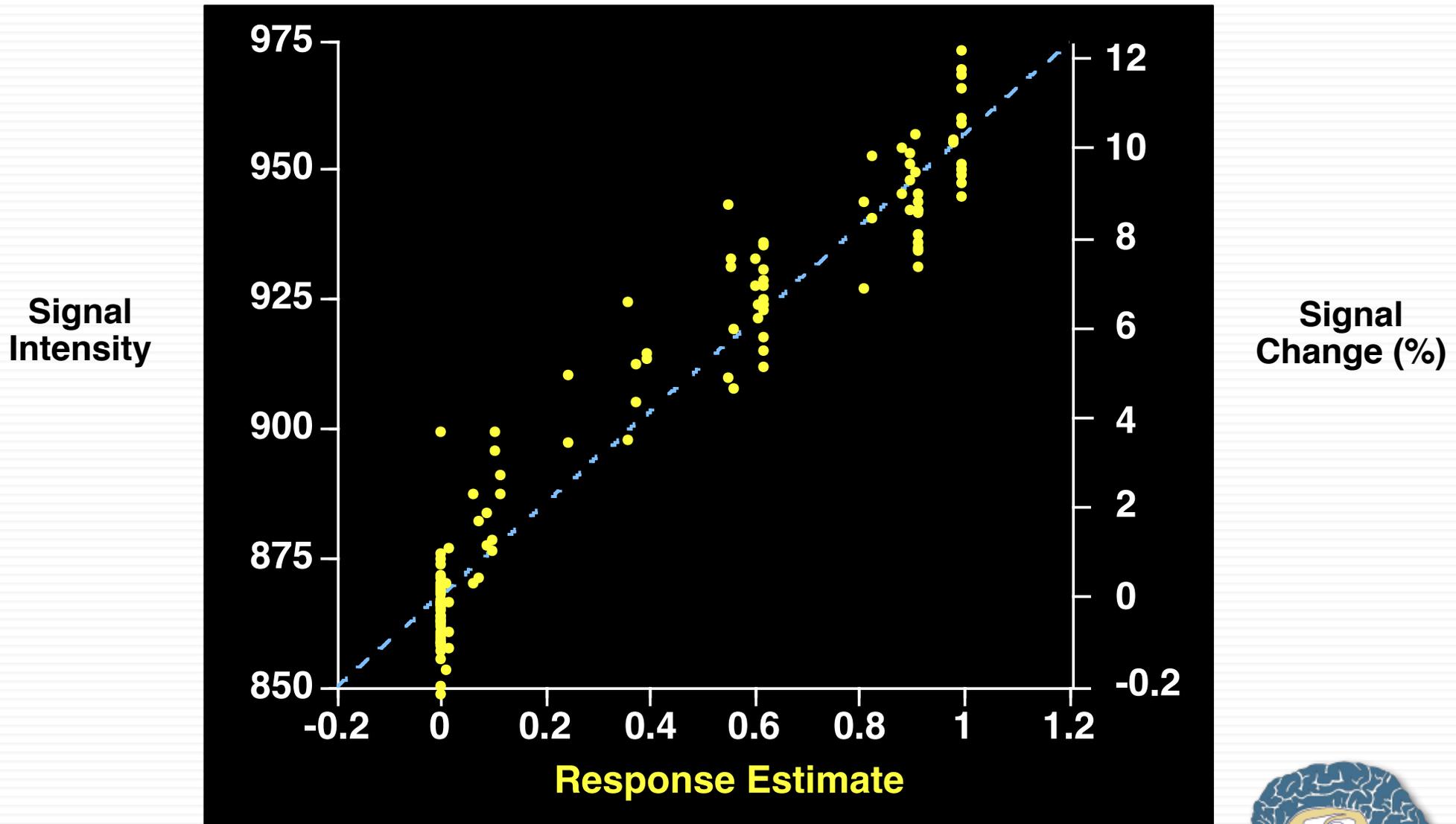
Ken Kwong



Amplitude-weighted Linear Estimate

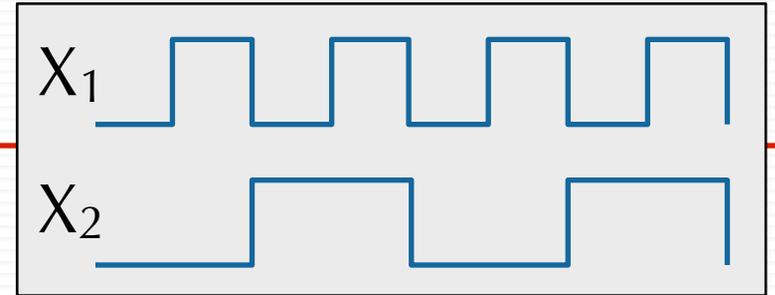


Estimated vs. Actual *f*MRI Response



General Linear Model

(matrix form)

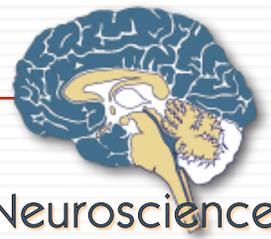


$$\mathbf{Y} = \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2 + \varepsilon$$

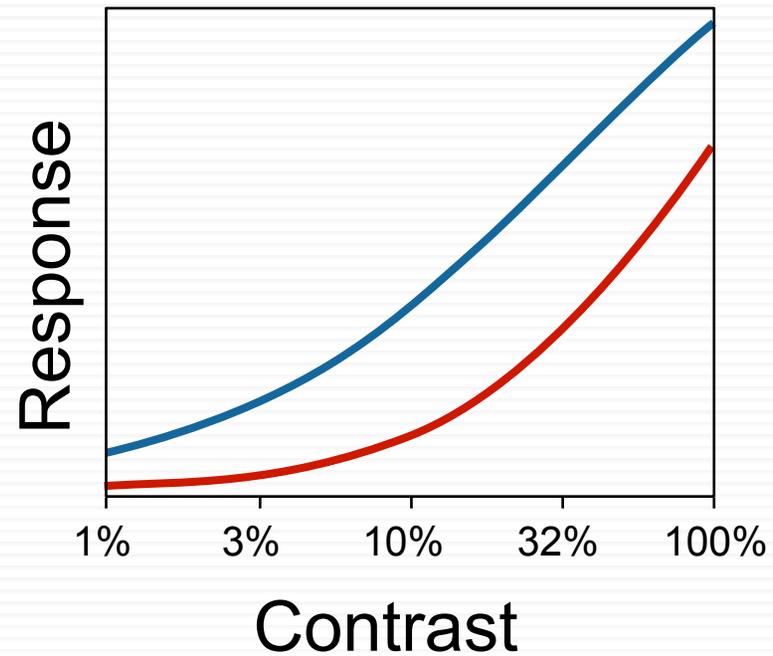
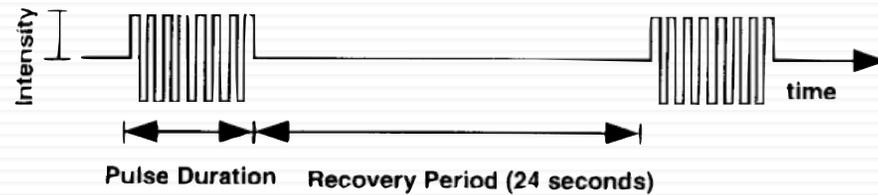
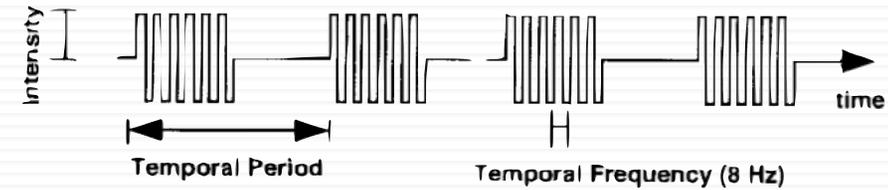
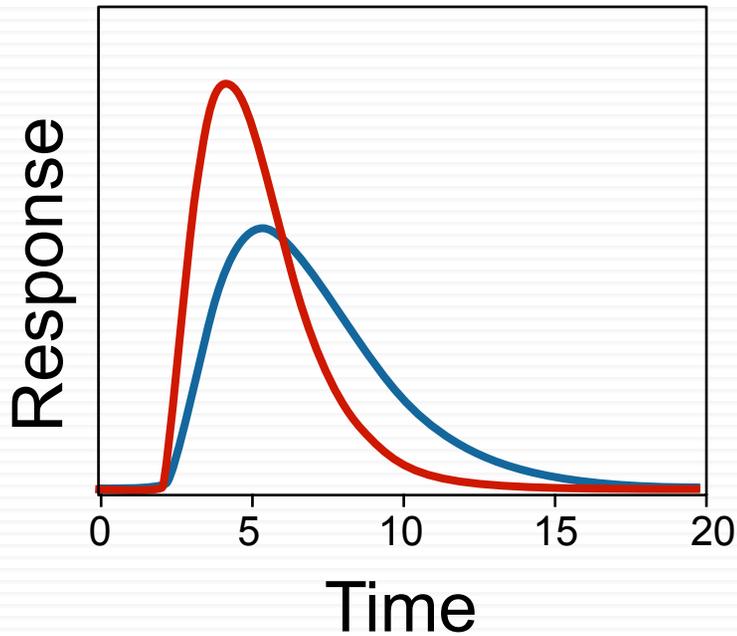
$$\mathbf{Y} = f(\mathbf{X}_1)\beta_1 + f(\mathbf{X}_2)\beta_2 + \varepsilon$$

$$\mathbf{Y} = \ln(\mathbf{X}_1)\beta_1 + \ln(\mathbf{X}_2)\beta_2 + \varepsilon$$

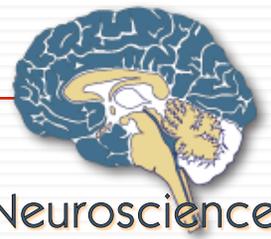
$$\mathbf{Y} = \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2 + \mathbf{X}_1\mathbf{X}_2\beta_3 + \varepsilon$$



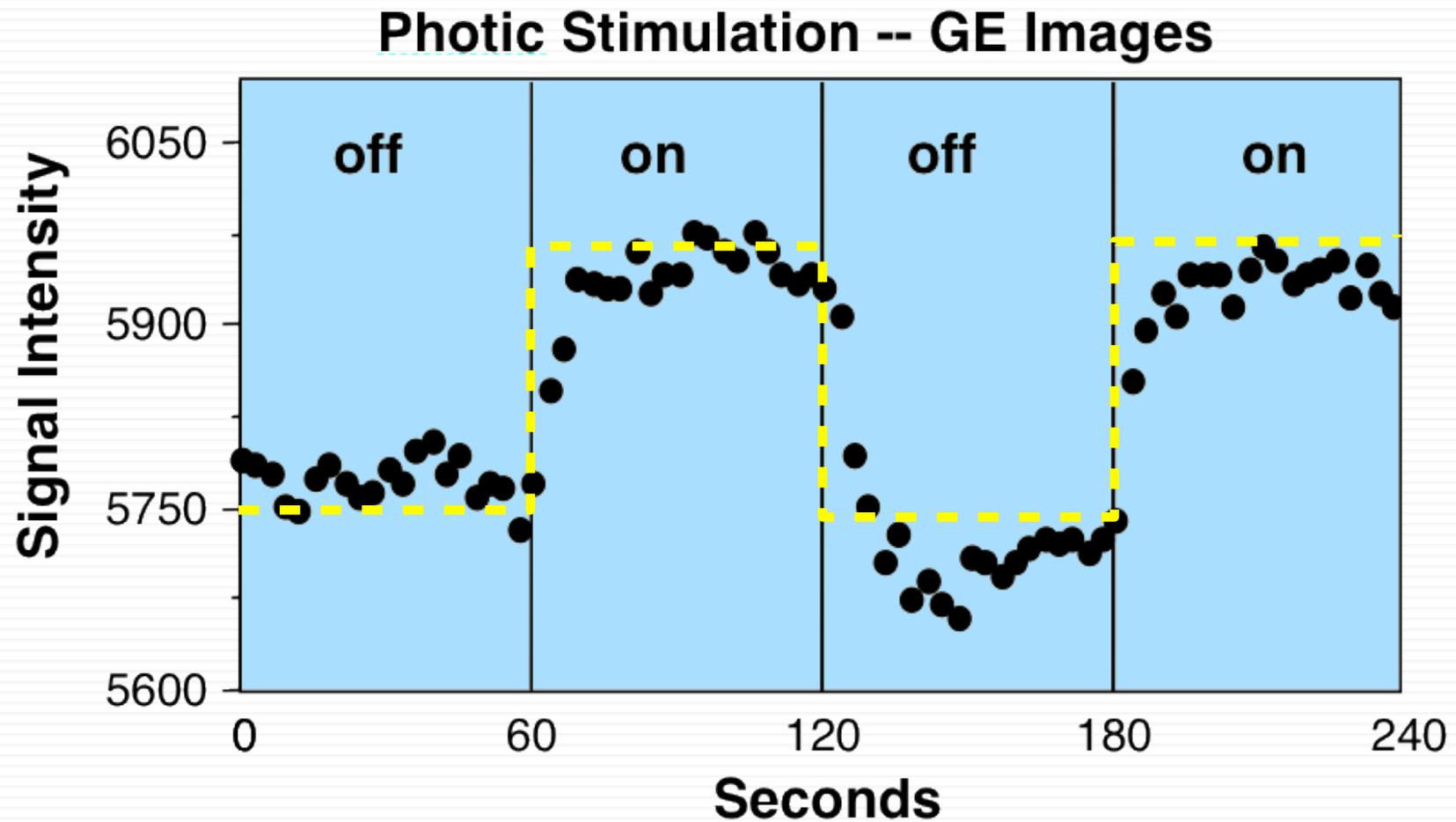
Time Invariance (redux)



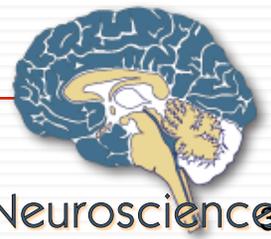
Boynton, et al., 1996



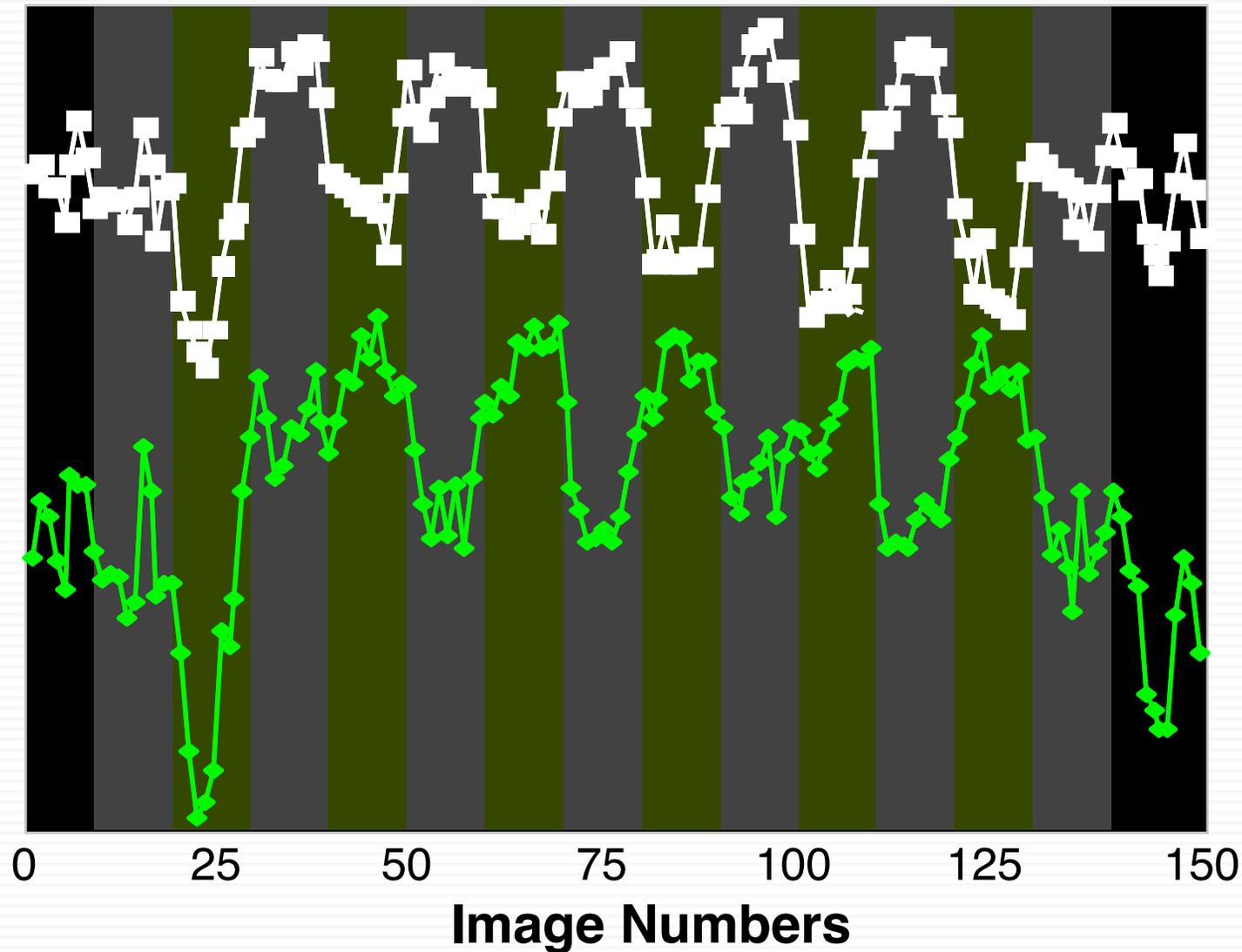
Gradient-Recalled Echo



Ken Kwong

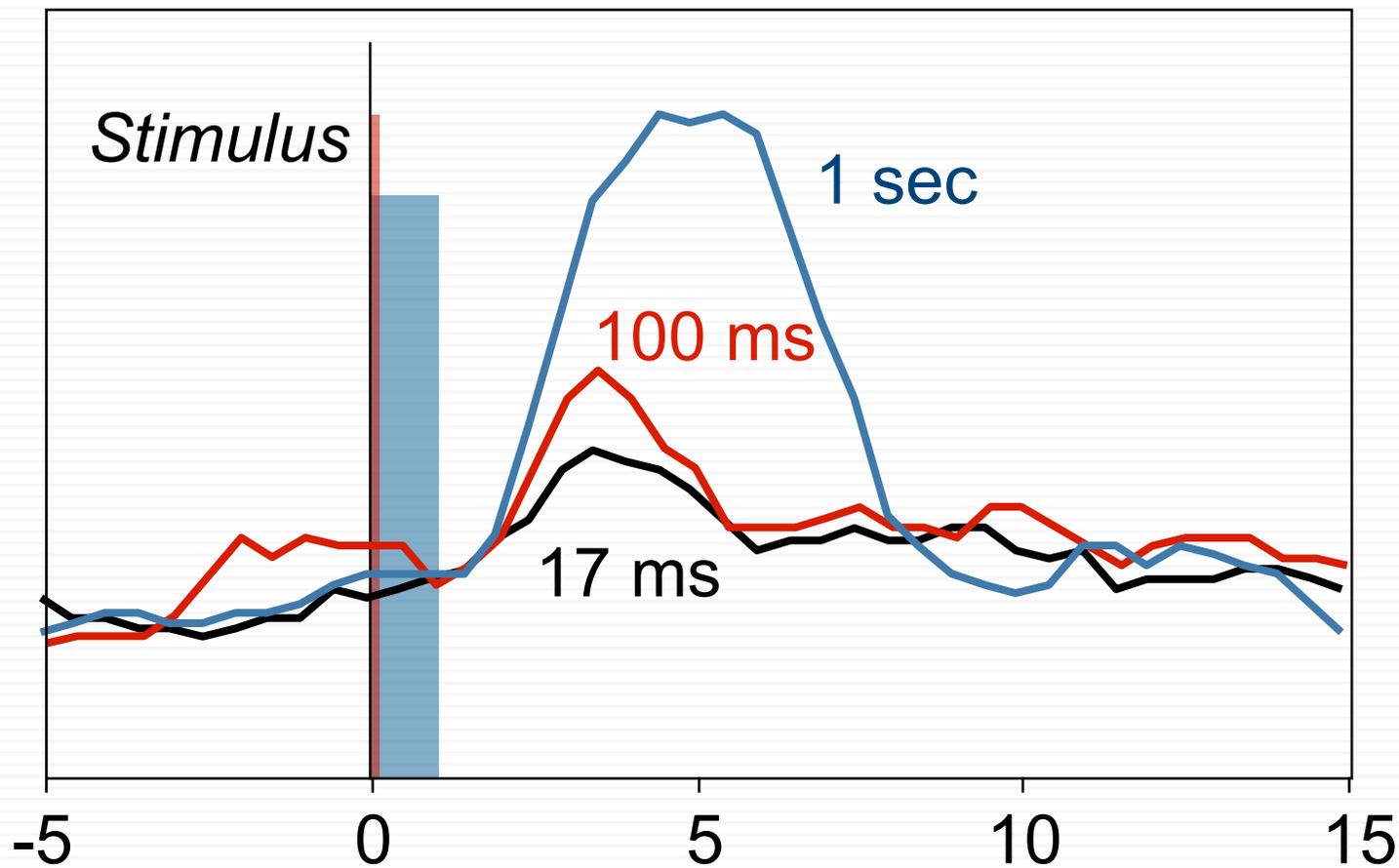


Hemifield Alternation 20 seconds



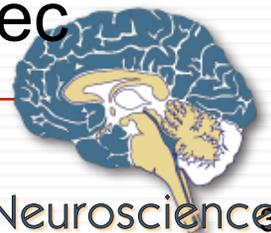
Response Latency vs. Stimulus Duration

Average of 10 recordings

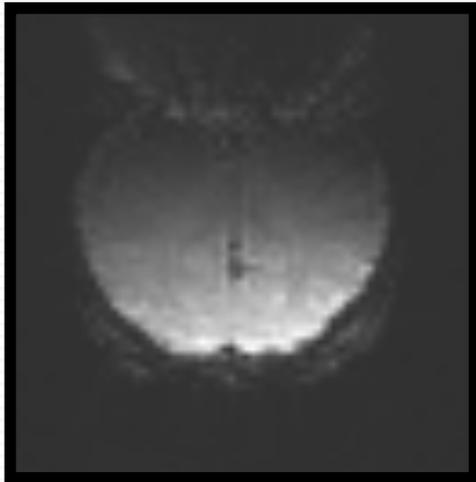


Data courtesy of Robert Savoy

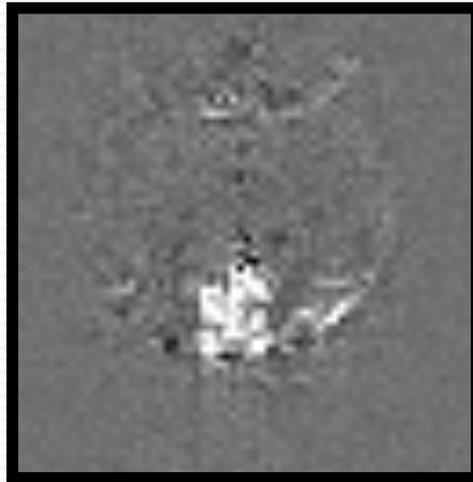
sec



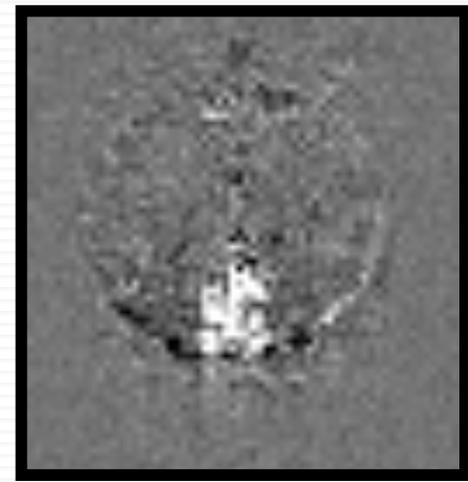
Binocular vs Monocular Activation



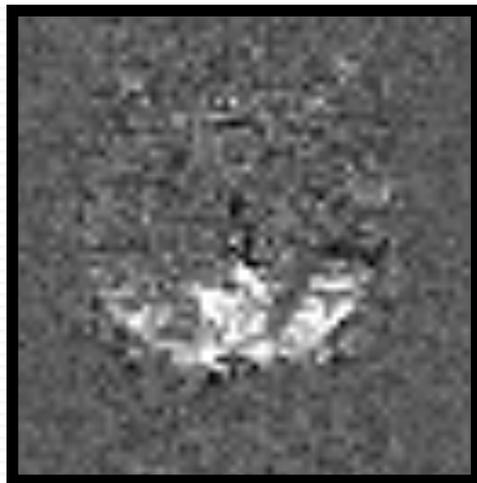
baseline



Binocular



Monocular



Bino minus Mono



Extrastriate activation



The Euler Relation

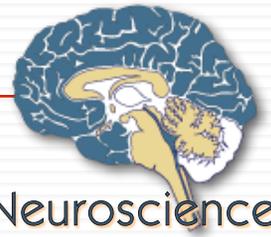
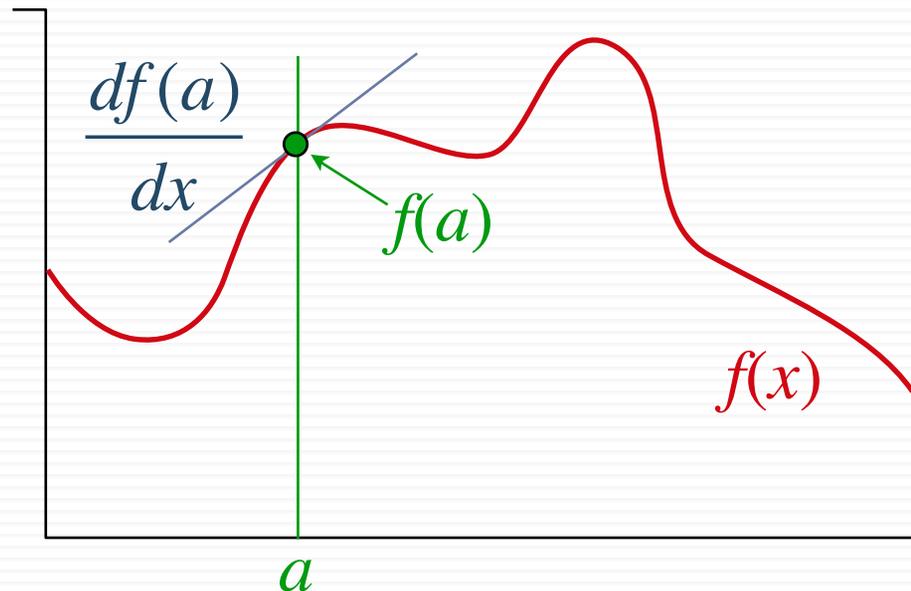
$$e^{ix} = \cos x + i \sin x$$

The Taylor Series

$$f(x) = f(a)(x-a)^0 + \frac{df(a)}{dx} \frac{(x-a)^1}{1!} + \frac{d^2 f(a)}{dx^2} \frac{(x-a)^2}{2!} + \dots + \frac{d^n f(a)}{dx^n} \frac{(x-a)^n}{n!} + \dots$$

$$= \sum_0^{\infty} f^{(n)} \frac{(x-a)^n}{n!}$$

$$\approx f(a) + (x-a) \frac{df}{dx}$$



The Euler Relation - getting there

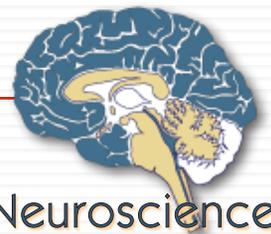
Taylor Series

$$f(x) = f(a)(x-a)^0 + \frac{df(a)}{dx} \frac{(x-a)^1}{1!} + \frac{d^2 f(a)}{dx^2} \frac{(x-a)^2}{2!} + \dots + \frac{d^n f(a)}{dx^n} \frac{(x-a)^n}{n!} + \dots$$
$$= \sum_0^{\infty} f^{(n)}(a) \frac{(x-a)^n}{n!}$$

McLaurin Series

$$f(x) = f(0) + \frac{df(0)x}{dx} + \frac{d^2 f(0) x^2}{dx^2 2!} + \frac{d^3 f(0) x^3}{dx^3 3!} + \dots + \frac{d^n f(0) x^n}{dx^n n!} + \dots$$

$$f(x) = \sum_{n=1}^{\infty} f^{(n)}(0) \frac{x^n}{n!}$$



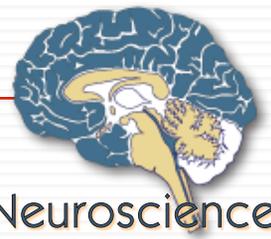
The Euler Relation - getting there

$$f(x) = \sum_{n=1}^{\infty} f^{(n)} \frac{x^n}{n!} + f(0)$$

$$\begin{aligned} \sin(x) &= \sin(0) + x \cos(0) - \frac{x^2 \sin(0)}{2!} - \frac{x^3 \cos(0)}{3!} + \frac{x^4 \sin(0)}{4!} + \frac{x^5 \cos(0)}{5!} + \dots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \end{aligned}$$

$$\begin{aligned} \cos(x) &= \cos(0) + x \sin(0) - \frac{x^2 \cos(0)}{2!} - \frac{x^3 \sin(0)}{3!} + \frac{x^4 \cos(0)}{4!} + \frac{x^5 \sin(0)}{5!} + \dots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \end{aligned}$$

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!} + \frac{u^6}{6!} + \frac{u^7}{7!} + \dots$$



The Euler Relation - getting there

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!} + \frac{u^6}{6!} + \frac{u^7}{7!} + \dots$$

$$e^{ix} = 1 + ix + \frac{i^2 x^2}{2!} + \frac{i^3 x^3}{3!} + \frac{i^4 x^4}{4!} + \frac{i^5 x^5}{5!} + \frac{i^6 x^6}{6!} + \frac{i^7 x^7}{7!} + \dots$$

$$= 1 + ix - \frac{x^2}{2!} - i \frac{x^3}{3!} + \frac{x^4}{4!} + i \frac{x^5}{5!} - \frac{x^6}{6!} - i \frac{x^7}{7!} + \dots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + i \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right]$$

$$= \cos(x) + i \sin(x).$$

Fourier transform

$$\begin{aligned}\mathcal{F}(s) &= \mathcal{F}(f(x)) \\ &= \int_{-\infty}^{\infty} f(x) [\cos(2\pi sx) - i \sin(2\pi sx)] dx \\ &= \int_{-\infty}^{\infty} f(x) e^{-2\pi i s x} dx\end{aligned}$$

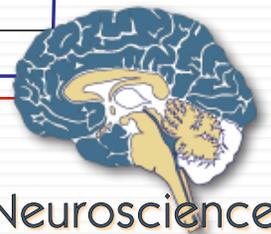
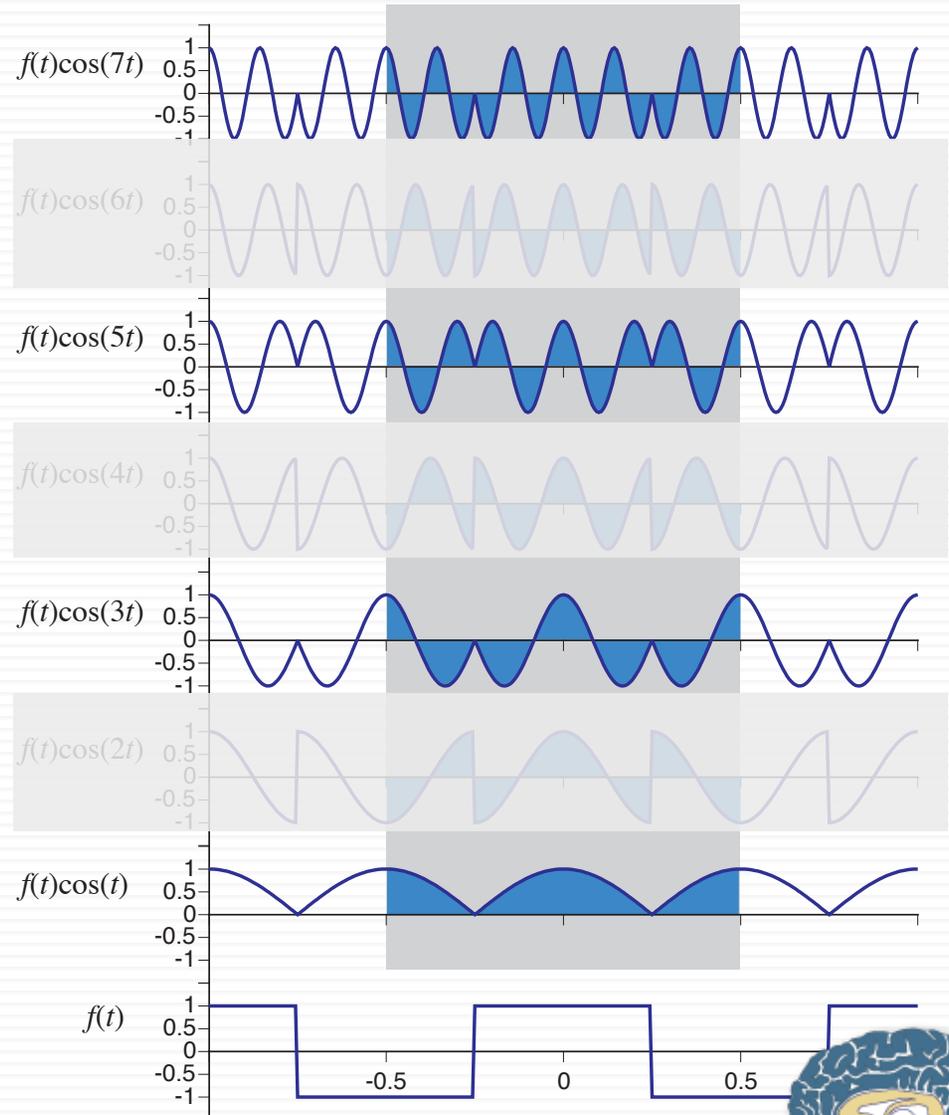
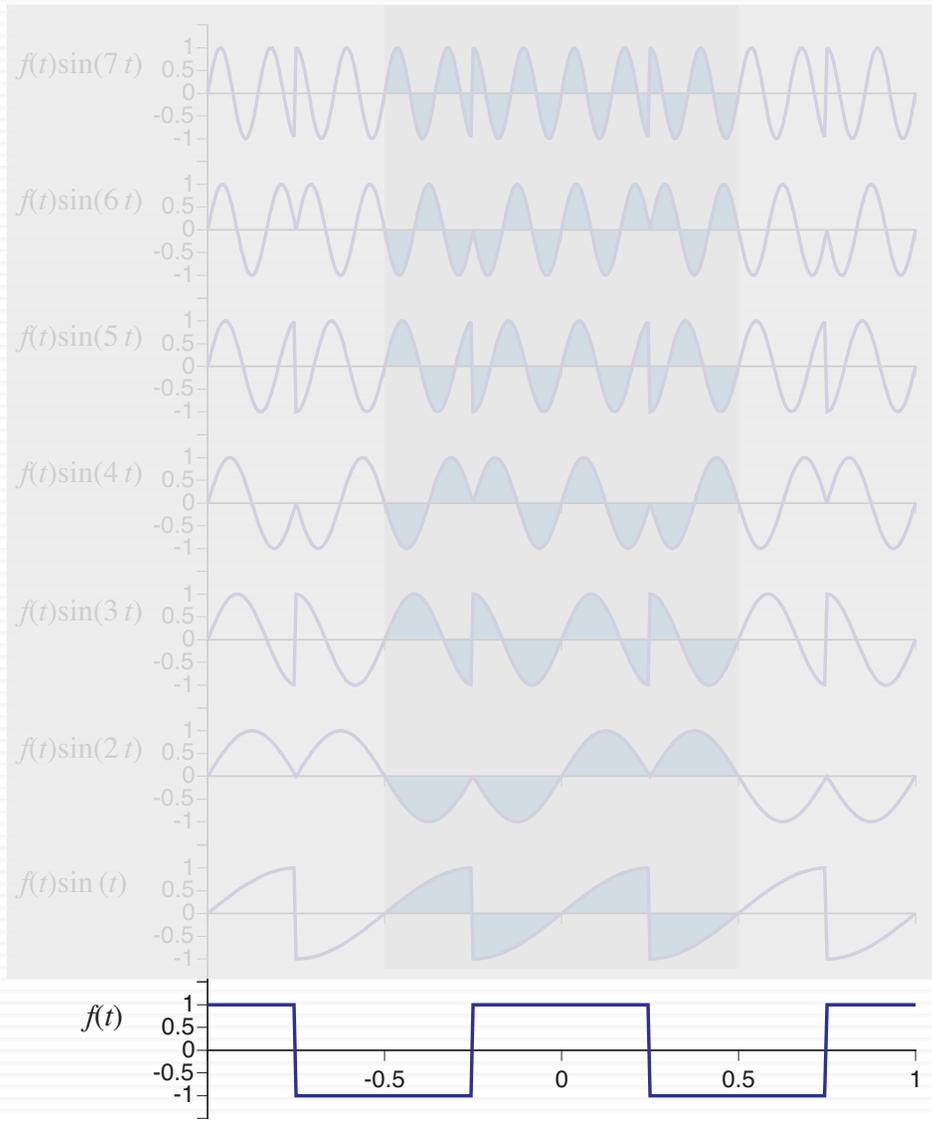
$$\begin{aligned}\mathcal{F}^{-1}(\mathcal{F}(s)) &= f(x) \\ &= \int_{-\infty}^{\infty} f(x) e^{+2\pi i s x} dx.\end{aligned}$$

Let $x = \frac{\omega}{2\pi}$:

$$\mathcal{F}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$\mathcal{F}^{-1}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) e^{2\pi i \omega x} d\omega.$$

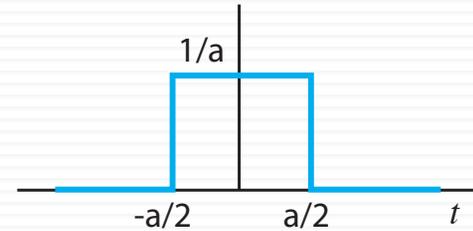
Fourier transform



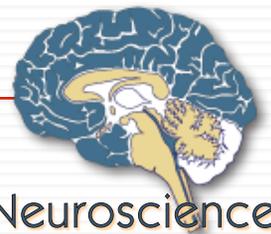
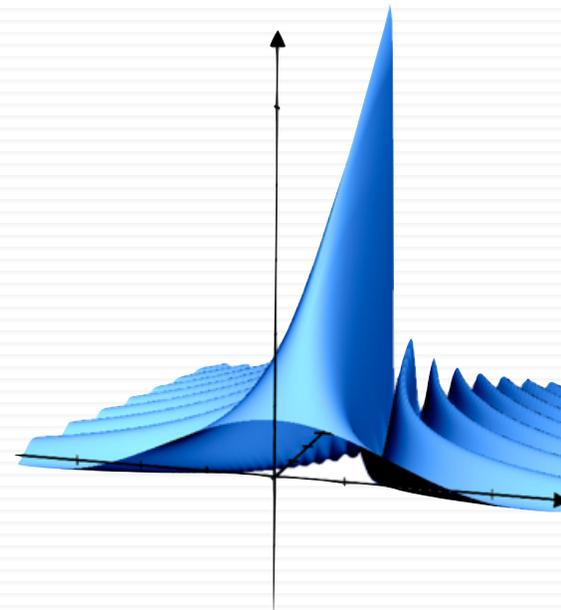
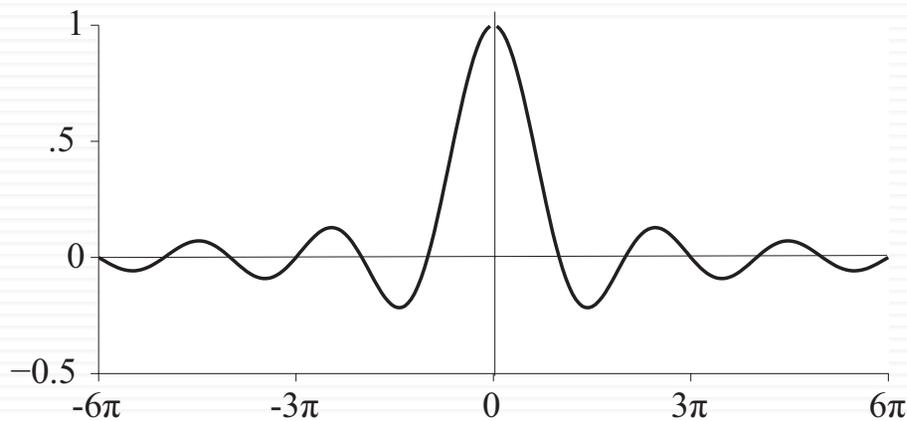
Using the Fourier Transform

■ Important Pairs - Square and sinc:

$$f(t) = \begin{cases} 1/a & \text{for } -a/2 < t < a/2 \\ 0 & \text{otherwise} \end{cases}$$



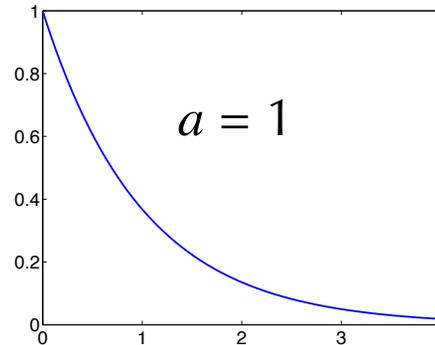
$$f(s) = \mathcal{F}(f(t)) = \frac{\sin(\pi a s)}{\pi a s}$$



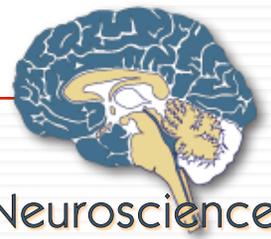
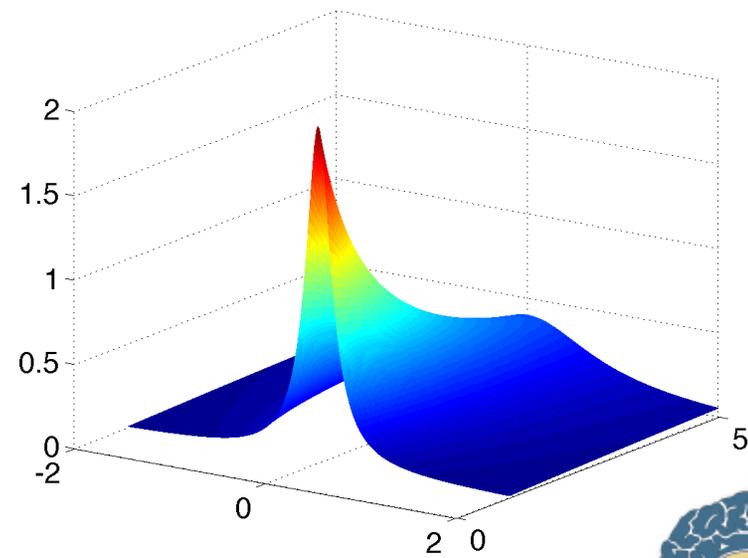
Using the Fourier Transform

■ Important Pairs - Exponential and Lorentzian:

$$f(t) = e^{-a|x|}$$



$$\mathcal{F}(s) = \frac{2a}{4\pi^2 s^2 + a^2}$$

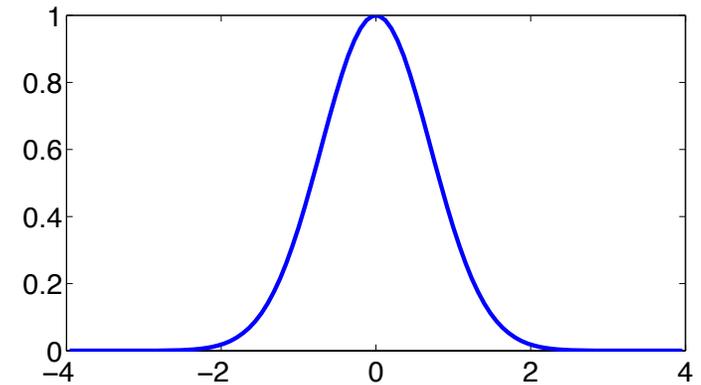


Using the Fourier Transform

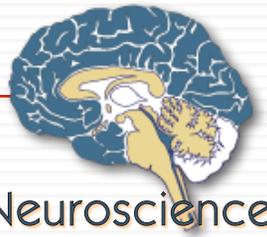
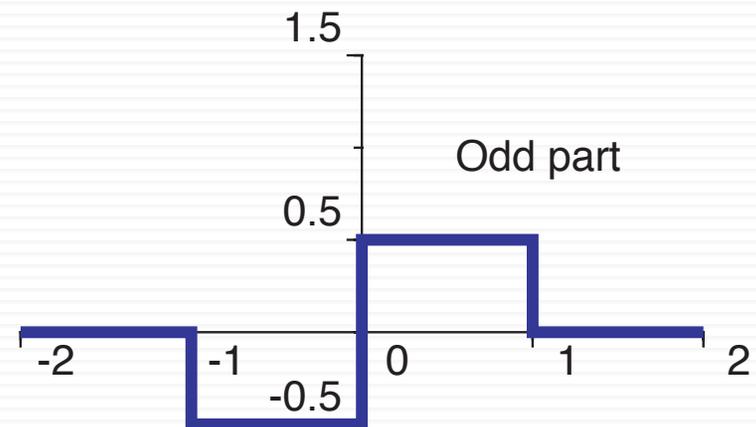
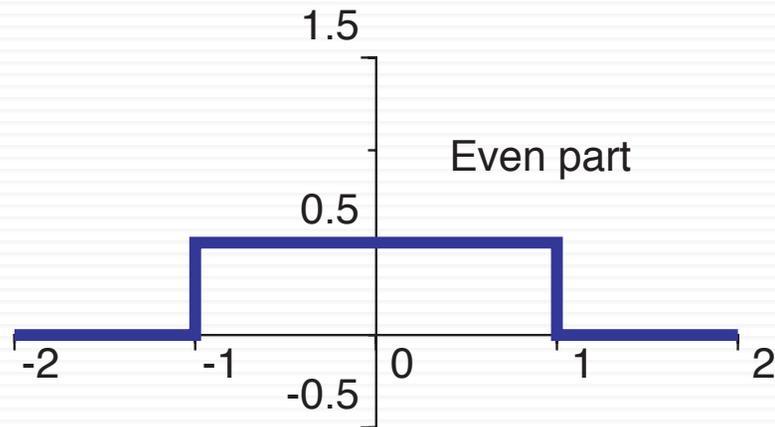
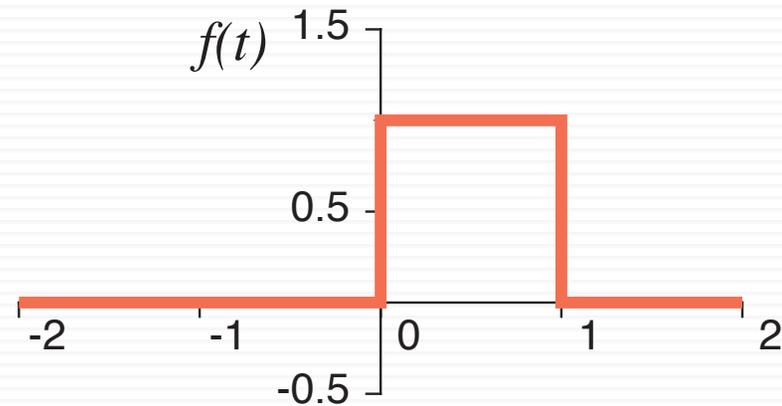
■ Important Pairs - Gaussian:

$$f(t) = e^{-ax^2}$$

$$\mathcal{F}(s) = \frac{\sqrt{\pi} e^{-\pi x^2}}{\sqrt{a}}.$$



Odd and Even



Odd and Even

if a function is:

Real and Even

Real and Odd

Imaginary and Even

Complex* and Even

Complex and Odd

Real and Asymmetrical

Imaginary and Asymmetrical

Real Even + Imaginary Odd

Real Odd + Imaginary Even

Even

Odd

its Fourier transform is:

Real and Even

Imaginary and Odd

Imaginary and Even

Complex and Even

Complex and Odd

Complex and Hermitian

Complex and anti-Hermitian

Real

Imaginary

Even

Odd



Shift Theorem

$$\begin{aligned}\mathcal{F}[f(t-a)](s) &= \int_{-\infty}^{\infty} f(t-a)e^{-2\pi ist} dt \\ &= e^{-2\pi isa} \int_{-\infty}^{\infty} f(t-a)e^{-2\pi ist} e^{2\pi isa} dt \\ &= e^{-2\pi isa} \int_{-\infty}^{\infty} f(t-a)e^{-2\pi is(t-a)} dt.\end{aligned}$$

Let $u = t-a$ and $du = dt$:

$$\begin{aligned}&= e^{-2\pi isa} \int_{-\infty}^{\infty} f(u)e^{-2\pi isu} du \\ &= e^{-2\pi isa} \mathbf{F}[f(t)](s) \\ &= (\cos(2\pi sa) - i \sin(2\pi sa))F[f(t)](s)\end{aligned}$$

Convolution Theorem

$$\mathcal{F}(f(x) \otimes g(x)) = \mathcal{F}(f(x))\mathcal{F}(g(x))$$

$$\mathcal{F}(f(x)g(x)) = \mathcal{F}(f(x)) \otimes \mathcal{F}(g(x)).$$